

Navier Stokes and Thermodynamics V0.93

Abstract

The objective of this short note is to deduce the thermodynamic equations for fluids and their coupling with the Navier-Stokes equations. Some simplifications are made during the modeling and they are explicitly stated. There are also comments when and why the simplifications are true.

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Introduction

The modeling of fluids is done from the basic laws of physics, particularly the conservations laws for momentum (Newton's second law), mass and energy. Although Einstein showed the equivalence of mass and energy, thus replacing two conservation laws by one, in classical fluids conversion of mass/energy is not relevant and two individual laws are used.

In addition to these three conservation laws we need an equation of state, relating several variables (pressure, temperature, density).

Conservation of Mass for Fluids

The deduction of this equation and of the material derivative is done in many books ([Bat] for e.g.) and here we only restate the equations.

If u is the velocity and ρ is the density of a fluid, the **conservation of mass** equation for a fluid is

$$\frac{D\rho}{Dt} + \rho \operatorname{div} u = 0$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$$

is the material derivative.

Remark: The incompressibility assumption is based on the following. Let ρ_0 be the basic density, $\Delta\rho$ the characteristic value of $\rho - \rho_0$, U the characteristic velocity and L the characteristic length. Then

$$\frac{D\rho}{Dt} \sim \frac{\Delta\rho}{t} + U \frac{\Delta\rho}{L}$$

Since $U = L/t$,

$$\frac{1}{\rho} \frac{D\rho}{Dt} \sim \frac{U}{L} \frac{\Delta\rho}{\rho_0}.$$

Also $\text{div } u \sim U/L$. Hence if $\Delta\rho/\rho_0 \ll 1$, the conservation of mass equation implies that $\text{div } u \sim 0$. This, in practice, is always satisfied for liquids. This is also true for fluids (including gases !) at low Mach numbers, for e.g. in fluids which are not strongly heated. It is certainly not valid for combustion problems. Notice that incompressibility ($\text{div } u = 0$) does not imply that the density is constant, only that it is constant along the fluid path. For example in ocean dynamics there exists a thermal stratification responsible for spatial density variations.

Conservation of Momentum for Fluids

From Newton's Law, $F = ma = m \cdot du/dt$. Hence if δm is the mass of a small fluid particle,

$$\delta m \frac{Du}{Dt} = [\text{body forces}] + [\text{surface forces}].$$

The body (external) forces are gravity, coriolis, Lorenz force in the case of an electrically-conducting fluid, etc. The modeling of the surface forces is complex (see for eg. [Zei] or [Bat]). In a newtonian fluid we assume that the stress tensor is linear with respect to the deformation tensor $e_j^i = 1/2(\partial u^i/\partial x_j + \partial u^j/\partial x_i)$ and isotropic. We will assume also that the dynamic viscosity μ is constant. It may vary with the physical (eg. temperature, pressure) properties of the fluid, as in gases in high temperature or in the Earth's outer mantle. Under these hypotheses, the **conservation of momentum** equation for a fluid is

$$\rho \frac{Du}{Dt} = \rho \vec{g} - 2\rho \vec{\Omega} \times u - \nabla p + \mu \left(\Delta u + \frac{1}{3} \nabla(\text{div } u) \right),$$

where p is the pressure, \vec{g} is the gravity, $\vec{\Omega}$ is the coriolis force.

Remark: The momentum equation without viscous and coriolis terms is called Euler equation.

Thermodynamic Equations for Fluids

The first law of thermodynamics applied to fluids is a statement of how the thermal energy is related to the work done by compression or expansion (i.e., by changing its volume). It is a law of conservation of energy, relating the mechanical energy, the energy lost due to shear stresses and the energy supplied to the system.

Let e be the internal energy (for a fluid $e = cT$, it is related to the mean velocity of its molecules) per unit mass of a small fluid particle with volume δV , mass δm and surface $\delta \Sigma$.

- a) The kinetic energy is $\delta m \cdot u^2/2$ (i.e., $mv^2/2$).
 b) The potential energy is $\delta m \cdot \Phi$ (i.e., mgh), where Φ is the geopotential, $\nabla\Phi = g$.

c) The internal energy is $\delta m \cdot e$.

Hence the total energy (e_{tot}) of this small fluid particle is

$$e_{\text{tot}} = \delta m(u^2/2 + \Phi + e) = \int_{\delta V} (u^2/2 + \Phi + e)\rho \, dV \quad (1)$$

since $\int_{\delta V} \rho \, dV = \delta m$.

The first law of thermodynamics (“energy is conserved”) implies that the variation of the total energy of this small fluid particle is the sum of heat (dq/dt) and power (dw/dt) transferred across its surface. Hence (see [Gra] pag. 217–222):

$$\frac{De_{\text{tot}}}{Dt} = \frac{dq}{dt} + \frac{dw}{dt}. \quad (2)$$

dq/dt is the net heat transferred across the surface (to and from the surroundings). It is the (algebraic) sum of the heat supplied to the system and the heat transferred by molecular diffusion across the surface. If Q is the rate of heat per unit mass supplied to the system (eg. chemical reactions, sun heating), we have

$$\frac{dq}{dt} = \delta m Q - [\text{mol. diff. heat loss}] = \int_{\delta V} \rho Q \, dV - [\text{mol. diff. heat loss}].$$

The molecular diffusion heat loss is computed using Fourier’s Law: “heat loss is proportional to the variation of temperature”. The heat loss at any point of the boundary is proportional to the variation of the temperature on this point. Hence heat loss = $-\lambda \cdot \partial T/\partial n$, where λ is the thermal diffusivity. So, using Stoke’s theorem,

$$[\text{mol. diff. heat loss}] = - \int_{\delta \Sigma} \lambda \frac{\partial T}{\partial n} \, d\Sigma = - \int_{\delta \Sigma} \lambda \nabla T \cdot n \, d\Sigma = - \int_{\delta V} \text{div}(\lambda \nabla T) \, dV.$$

Hence we get

$$\frac{dq}{dt} = \int_{\delta V} (\rho Q + \text{div}(\lambda \nabla T)) \, dV. \quad (3)$$

dw/dt is the net power transferred across the surface (to and from the surroundings). The power can be due to mechanical means (shaft for eg.), by viscous (tangential) stresses or by normal stresses (pressure). If we desconsider the power due to mechanical means and write P_e as the work done by surface (tangential and normal) stresses,

$$\frac{dw}{dt} = P_e. \quad (4)$$

Putting together (2), (3) and (4) we get

$$\frac{De_{\text{tot}}}{Dt} = \int_{\delta V} (\rho Q + \text{div}(\lambda \nabla T)) dV + P_e. \quad (5)$$

Integrating the momentum equation against ρu we get (see [Les] pag. 24 for details)

$$\frac{D}{Dt} \int_{\delta V} (u^2/2 + \Phi) \rho dV = P_e + \int_{\delta V} (p \text{div } u - \mu \chi) dV,$$

where $\chi = 2(e_j^i e_i^j - (\text{div } u)^2/3)$. Using (1) and this relation we get

$$\frac{De_{\text{tot}}}{Dt} = P_e + \int_{\delta V} \left(\rho \frac{De}{Dt} + p \text{div } u - \mu \chi \right) dV. \quad (6)$$

If we equate (5) and (6) together the term P_e gets canceled out. Letting δV goes to zero we finally get

$$\rho \frac{De}{Dt} = \rho Q + \text{div}(\lambda \nabla T) - p \text{div } u + \mu \chi.$$

The second law of thermodynamics (“the entropy always increases”) implies that λ and μ are positive (see [DL] pag. 22 note 26).

Assuming that λ is a constant and using the conservation of mass equation, we can write this as (see [Ped] pag. 11)

$$\rho \frac{De}{Dt} = \rho Q + \lambda \Delta T + \frac{p}{\rho} \frac{D\rho}{Dt} + \mu \chi$$

For fluids the internal energy is, approximately, proportional to the temperature. Assuming that $e = cT$ and dividing by ρ the last equation, we get the **conservation of energy** equation for a fluid

$$c \frac{DT}{Dt} = Q + \frac{\lambda}{\rho} \Delta T - p \frac{D}{Dt} \left(\frac{1}{\rho} \right) + \frac{\mu}{\rho} \chi. \quad (7)$$

$\mu \chi$ is the heating coming from internal molecular viscous dissipation of kinetic energy. It is very small except when strong compressibility effects exists. It should be considered when we are modeling hypersonic gases or the slow circulation of the earth’s mantle. Hence hereafter we will assume that $\mu = 0$.

Remark: An easy computation shows that

$$\chi = 2|\nabla u|^2 + (u_y^1 + u_x^2)^2 - \frac{2}{3}(\text{div } u)^2 \text{ in } R^2 \text{ and}$$

$$\chi = 2|\nabla u|^2 + (u_y^1 + u_x^2)^2 + (u_z^1 + u_x^3)^2 + (u_z^2 + u_y^3)^2 - \frac{2}{3}(\text{div } u)^2 \text{ in } R^3.$$

Remark: The reciprocal of density is termed specific volume. If we write $V = 1/\rho$, ignore heat diffusion ($\lambda = 0$) and assume there is no addition or subtraction

of heat ($Q = 0$) then (7) becomes $c \cdot DT/Dt = -p \cdot DV/Dt$. We see clearly from this equation that if a fluid is compressed then its temperature will increase and if it is expanded it will cool.

A Fluid can be classified as:

a) liquid: incompressible fluid (water for example).

b) gas: compressible fluid (air for example).

For a liquid $e = c_p T$ and for a gas $e = c_v T$ where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume. They are related by $R = c_p - c_v$, where R is the gas constant for dry air.

For a **liquid** ρ and T are linearly related. Hence $\rho = \rho_0(1 - \alpha_T(T - T_0))$. Using this and $c = c_p$ in (7), after some manipulation, we get

$$\left(c_p \rho + \frac{p \rho_0 \alpha_T}{\rho} \right) \frac{DT}{Dt} = \rho Q + \lambda \Delta T$$

and

$$\left(\frac{\rho}{\rho_0 \alpha_T} - p \right) c_p \frac{D\rho}{Dt} = -\rho Q + \frac{\lambda}{\rho_0 \alpha_T} \Delta \rho.$$

Remark: If $Q = \lambda = 0$ (an adiabatic process, i.e., no heat transfer) then from this equation, $D\rho/Dt = 0$, which implies from the conservation of mass equation, $\text{div } u = 0$. Hence an adiabatic liquid is incompressible.

Assuming incompressibility ($\text{div } u = 0$),

$$c_p \frac{DT}{Dt} = Q + \frac{\lambda}{\rho} \Delta T$$

and

$$c_p \frac{D\rho}{Dt} = -Q \rho_0 \alpha_T + \frac{\lambda}{\rho} \Delta \rho.$$

So this simplification does not prevent variation of the density which will affect the velocity through the momentum equation.

For a **gas** we have the state equation $p = RT\rho$. Using this and $c = c_v$ in (7) we get

$$c_v \frac{DT}{Dt} + R\rho \frac{D}{Dt} \left(\frac{T}{p} \right) = Q + \frac{\lambda}{\rho} \Delta T.$$

Expanding this and using $R = c_p - c_v$, we get (see [WP] pag. 70)

$$c_p \frac{DT}{Dt} - \frac{RT}{p} \frac{Dp}{Dt} = Q + \frac{\lambda}{\rho} \Delta T.$$

Boussinesq Approximation

This approximation is done for an incompressible fluid. The hydrostatic equilibrium is found from the momentum equation by letting $u = 0$: $\nabla\bar{p} = \bar{\rho}\vec{g}$. So \vec{g} is normal to the isobaric surfaces. Taking the curl we conclude also that \vec{g} is normal to the iso-density surfaces.

The essence of the Boussinesq approximation is to assume that the pressure and density are close to these basic profiles, i.e., $p = \bar{p}(x) + p'(x, t)$ and $\rho = \bar{\rho}(x) + \rho'(x, t)$ with $|p'| \ll \bar{p}$ and $|\rho'| \ll \bar{\rho}$. Assuming also that the thickness of the layer is small with respect to the total depth of the fluid, so that $\bar{\rho}$ is close to ρ_0 , we get

$$\rho_0 \frac{Du}{Dt} = \rho' \vec{g} - 2\rho_0 \vec{\Omega} \times u - \nabla p' + \mu \Delta u$$

and

$$\frac{D\rho'}{Dt} + u \cdot \nabla \bar{\rho} = k \Delta \rho'.$$

Hence the influence of the density on the velocity of the fluid is through the gravity (buoyancy force) term only.

The **Bénard problem** is basically this problem without the coriolis force. It can be deduced directly from the momentum equation by assuming that the density is a constant ρ_0 except for the buoyancy force. Also incorporating the constants in \tilde{Q} and $\tilde{\lambda}$,

$$\begin{aligned} \rho_0 \frac{Du}{Dt} &= \rho \vec{g} - \nabla p + \mu \Delta u, \\ \frac{D\rho}{Dt} &= \tilde{Q} + \tilde{\lambda} \Delta \rho. \end{aligned}$$

If we assume that the mean profile is uniform we get Navier-Stokes with constant density in the momentum equation and the diffusion equation above for the density. The density behaves as a passive scalar. This describes very well non-heated (or slightly-heated) fluids. The temperature is only a passive quantity which marks the flow, as would a dye. Another passive scalar that can be considered is the concentration of a chemical species in a river.

Remark: We have another approximation called the Barré de Saint-Venant equations, also know as the **shallow-water equations**. See [Les] pag. 43.

References:

The main reference for this text is [Les].

[Bat] Batchelor, G. K.; An Introduction to Fluid Mechanics; Cambridge University Press.

[DL] Dautray, Robert; Lions, Jacques-Louis; Mathematical Analysis and Numerical Methods fro Science and Technology; Vol.1; Springer-Verlag.

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[Zei] Zeidler, Eberhard; Nonlinear Functional Analysis and its Applications IV: Applications to Mathematical Physics; Springer-Verlag.