Exercises in PDE – Part IV

Version V0.33 May 2007
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1 First Order Equation: Characteristics

1.1 Characteristics

1. ([GL] p.22) Solve $xu_x+(x+y)u_y = u+1$, $u(x,0) = x^2$.

Answer: $u = x^2\exp(-y/x) + \exp(y/x) - 1$

2. ([IU] Fall 1980)

(a) Find the solution of the problem $u_x + u_y = u$, $u(x,x) = 1$ for $x \in \mathbb{R}$ and $y > x$ if it exists.

(b) Find the solution of the problem $x^2u_x + u_y = u$, $u(x,0) = f(x)$ for $x \in \mathbb{R}$, $y > 0$ and $f \in C^1$ if it exists. Explain carefully wheter the solution exists for all $x \in \mathbb{R}$, $y > 0$.

3. ([IU] Fall 1981, [Ga] p. 22, #2, #3)

(a) Find the solution, if it exists, of the problem $u_x+u_y = u^2$ passing through the curve $(x,y,u) = (t,-t,t)$. Is this solution global?

Hint: It becomes infinite along the hyperbola $x^2 - y^2 = 4$.

(b) Solve the problem $u_x + u_y = u$, passing through the curve $(x,y,u) = (t,t,1)$ if this exists.

Answer: There are no solutions.

4. ([GL] p.24, #4, #5) Solve the equations below for $u(x,0) = e^x$:

(a) $xu_x + u_y = 1$.

(b) $xu_x + (y^2+1)u_y = u$.

5. ([IU] Fall 1987) Solve the Cauchy problem $xu_x + yuu_y = -xy$ for $x > 0$, $u = 5$ on $xy = 1$, if a solution exists.

6. Solve in $\mathbb{R}^2$ $xu_x + yu_y = u+1$ with

(a) ([IU] Fall 1989, [GL] p. 23) $u(x,x^2) = x^2$.

Answer: $u = y + x^2/y - 1$

(b) ([GL] p.23) $u(x,x) = x^2$.

Answer: no solution

7. ([IU] Fall 1990) Find explicitly u solving the problem $uu_x + u_y = u$, $u(x,1) = 2x$.

8. ([IU] Fall 1993)

(a) Given a smooth $f : \mathbb{R} \to \mathbb{R}$ find the solution in implicit form to $uu_x+u_y = 0$ and $u(x,0) = f(x)$. Also describe the characteristics for this problem.

(b) Now suppose $f' < 0$ on some interval. Argue geometrically or analytically that no classical solution can exists for all $y > 0$.

9. ([IU] Winter 1991) Solve $xu_x - yu_y = x^2y^2$, $u(1,y) = 3$. Sketch the characteristcs in the xy-plane.

10. ([Ga] p.15) Prove that the general solution of

$$u_x = \frac{u_y}{1-u}$$

is given by $u(x,y) = \phi(x/(1-u)+y)$ where ϕ is an arbitrary function of one variable.

11. ([TZ] p.123, #2.1) Find the characteristic curves of each of the following operators in $\mathbb{R}^2$:

(a) $P(x,D) = D_1^2 + x_2D_2^2$;

(b) $P(x,D) = D_1^2 + x_2^2D_2^2$;

(c) $P(D) = D_1^4 + 2D_1^2D_2^2 + D_2^4$ (biharmonic operator);

(d) $P(x,D) = x_2^2D_1^2 - 2x_1x_2D_1D_2 + x_1^2D_2^2 + x_2D_1 + x_1D_2$.

Answer: ((a) $x_1 \pm 2(-x_2)^{1/2} = c$, $x_2 < 0$; (d) $x_1^2 + x_2^2 = c$

([TZ] p.123, #2.2) Find the characteristic curves of each of the following equations:

(a) $y^3u_{xx}+u_{yy} = 0$; (b) $u_x+2xyu_y+e^xu = \cos(x+y)$.

Answer: (a) $x = \pm 2/5(-y)^{5/2} + c$, $y < 0$; (b) $y = ce^{x^2}$,

12. ([TZ] p.136) Solve $u_x + xu_y = y$ with initial conditions:

(a) $u(0,y) = y^2$; (b) $u(0,y) = \sin(y)$; (c) $u(1,y) = 2y$.

Answer: general solution is $u(x,y) = xy - x^3/3 + f(y - x^2/2)$; (a) $u(x,y) = xy - x^3/3 + (y - x^2/2)^2$; (b) $u(x,y) = xy - x^3/3 + \sin(y - x^2/2)$; (c) $u(x,y) = xy - x^3/3 + y - x^2/2 + 5/6$;

13. ([Ha] p.422, #11.2.2, 11.2.5, 11.2.8) Solve using the method of characteristics:

(a) $w_t - 3w_x = 0$ with $w(x,0) = \cos(x)$ 11.2.2

(b) $w_t + tw_x = 1$ with $w(x,0) = f(x)$ 11.2.5 (b)

(c) $w_t + 3tw_x = w$ with $w(x,0) = f(x)$ 11.2.5 (d)

Answer: (a) $w(x,t) = \cos(x+3t)$; (b)$w(x,t) = t + f(xe^{-t})$; (c) $w(x,t) = f(x - 3t^2/2)e^t$.

1.2 Quasi-Linear Equations

14. ([Jo] p.18–19) Cap 1.6: General First-Order Quasi-Linear Equations 2 variables: Exercises: p. 18 - 19

15. ([TZ] p.61 and p. 62 #2.1) For each equation find the general integral:

(a) $xz_x + yz_y = z$.

Answer: $F(y/x, z/x) = 0$.

(b) $zz_x + yz_y = x$

Answer: $F(z^2 - x^2, (x+y)^2 - (y+z)^2) = 0$.

(c) $xz_x + yz_y = xy(z^2+1)$

Answer: $z = \tan(xy/2 + f(x/y))$.

(d) $x(y-z)z_x + y(z-x)z_y = z(x-y)$

Answer: $F(x+y+z, xyz) = 0$.

(e) $zz_y = -y$

Answer: $z^2 + y^2 = f(x)$.

16. ([TZ] p.67 and p. 68 #3.1) Solve the following initial value problems:

(a) $(y+z)z_x + yz_y = x - y$; $z = 1 + x$ on γ given by $y = 1$.

Answer: $z = 2/y + x - y$

(a) $z_x + z_y = z$; $z = \cos(t)$ on $\gamma : (t, 0)$;

Answer: $z = e^y \cos(x-y)$

(b) $x^2 z_x + y^2 z_y = z^2$; $z = 1$ on γ given by $y = 2x$;

Answer: $z = xy/(xy + 2x - y)$

(c) $x(y-z)z_x + y(z-x)z_y = z(x-y)$; $z = t$ on $\gamma : (t, 2t/(t^2-1)$ for $0 < t < 1$;

Answer: $z = (x+y)/(xy-1)$

(e) $yz_x - xz_y = 2xyz$; $z = t^2$ on $\gamma : (t,t)\ t > 0$;

Answer: $z = e^{(x^2-y^2)/2}(x^2+y^2)/2$

(g) $zz_x + yz_y = x$; $z = 2t$ on $\gamma : (t, 1)$.

Answer: $z = x(3y^2+1)/(3y^2-1)$

17. ([TZ] p.71, #4.1) Consider $zz_x + yz_y = x$ with initial data on $\gamma : (t,t),\ t > 0$. Decide wether there is a unique solution, no solution or infinitely many solutions in a neighborrhood of $(1,1)$ for the following initial data:

(a) $z = 2t$ on γ;

(b) $z = t$ on γ;

(c) $z = \sin(\pi t/2)$ on γ.

Answer: (a) unique; (b) infinitely many; (c) no solution.

18. ([TZ] p.74 and p. 75, #5.5) Solve $zz_x + z_y = 0$ with initial data:

(a) $z(x,0) = -x$; (b) $z(x,0) = x$.

Answer: (a) $z = x/(y-1)$ for $y < 1$; (b) $z = x/(y+1)$. No shocks here for $y = 1$.