Exercises in PDE – Part III

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1 Sobolev

1.1 Basics

1. ([Ev] Sec 2.5, #1) Prove that the space of Hölder continuous function is Banach.

2. ([Th] p.157, #9) Prove that ∂^α is a continuous operator from $W^{m,p}(\Omega) \to W^{m-|\alpha|,p}(\Omega)$.

3. ([Th] p.157, #9) Let Ω_1 and Ω open with $\Omega_1 \subset \Omega$. Define $T : W^{m,p}(\Omega) \to W^{m,p}(\Omega_1)$ by $T(u)(x) = u(x)$ for $x \in \Omega_1$. Prove that T (restriction operator) is a linear continuous operator.

4. Let $u, v \in W^1(\Omega)$. Prove that $uv \in W^1(\Omega)$ and $D(uv) = uDv + vDu$.

Hint: Use molifiers.

5. ([Jo] p.199, #4) Prove that if a sequence f_k converges in $W^{m,p}$ to f, then T_{f_k} converge in the sense of distributions to T_f.

6. ([Ev] Sec 2.5, #17) Suppose $u \in W^{1,p}(\Omega)$. Prove that:

(a) $|u| \in W^{1,p}(\Omega)$;

(b) $D|u| = Du$ a.e. for $u \geq 0$ and $D|u| = -Du$ a.e. for $u < 0$.

7. ([GT] p.174, #7.6) Let $\Omega \subset \mathbb{R}^n$ be a set containing the origin. Show that $|x|^{-\alpha}$ belongs to $W^k(\Omega)$ provided $k + \alpha < n$.

8. ([Fr] p.19, #2) Let B be the unit ball in $\mathbb{R}^n$ and $1 < p < \infty$. Find the largest integer k such that $|x| \in W^{k,p}(B)$.

Answer: $k < n$.

9. ([Ev] Sec 2.5, #13) Verify that $u(x) = \log\log(1+1/|x|)$ belongs to $W^{1,n}(0,1)$.

10. ([Ev] Sec 2.5, #16) (chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1 with F' bounded. Suppose Ω is open and bounded and $u \in W^{1,p}(\Omega)$ for $1 < p < \infty$. Show that $v = F(u) \in W^{1,p}(\Omega)$ and $v_{x_i} = F'(u)u_{x_i}$.

11. ([GT] p.173, #7.1) Let $\Omega \subset \mathbb{R}^n$ bounded. If $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$, we define

$$\Phi_p(u) = (|\Omega|^{-1} \int_\Omega |u|^p)^{1/p}.$$

Show that:

(a) $\lim_{p\to\infty} \Phi_p(u) = \sup_\Omega |u|$;

(b) $\lim_{p\to-\infty} \Phi_p(u) = \inf_\Omega |u|$.

12. ([Ev] Sec 2.5, #5, #6) Suppose $u \in W^{1,p}(0,1)$ ($p < \infty$). Prove that:

(a) u is equal a.e. to an absolutely continuous function;

(b) $|u(x) - u(y)| \leq |x-y|^{1-1/p}\|u'\|_{L^p}$ for a.e. $x, y \in [0,1]$.

Hint: Assume u is smooth and prove (b). Use density argument.

13. ([Ev] Sec 2.5, #10) Suppose Ω is connected and $u \in W^{1,p}(\Omega)$ satisfies $Du = 0$ a.e. in Ω. Prove that u is constant a.e. in Ω.

14. ([Fr] p.16, #3) Let $g \in W^{1,1}(a,b)$ and assume that $g(t) \geq 0$ and $g'(t) \leq 0$ almost everywhere, and that $g(t) \equiv 0$ in some interval (a,c) with $a < c < b$. Prove that $g(t) = 0$ almost everywhere in (a,b).

15. ([Io] p.356, #19) If $f, \Delta f \in L^2(\mathbb{R}^n)$, then $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq 2$.

Hint: Use Fourier transform and $0 \leq (1-|a|)^2$ e $(0 \leq (1-(|a|-|b|))^2$. The laplacian in L^2 implies the existence of all derivatives of lower order.

16. ([Ta] p.275, #9) Suppose that $P(D)$ is an elliptic differential operator of order m, i.e., $|P(\xi)| \geq C|\xi|^m$ for $|\xi|$ large. If $a < s+m$, show that $u \in H^a(\mathbb{R}^n), p(D)u \in H^s(\mathbb{R}^n)$ implies that $u \in H^{s+m}(\mathbb{R}^n)$.

Hint: Use Fourier transform. Let $k = (1+|\xi|^2)^{1/2}$ and estimate k^{s+m} in terms of k^a and $k^s P(\xi)$.

1.2 Extension/Trace

17. ([Ev] Sec 2.5, #14) Show that "a typical" function $u \in L^p(\Omega)$ does not have a trace on $\partial\Omega$. More precisely, prove there does not exists a bounded linear operator $T : L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ whenever $u \in C(\bar{\Omega}) \cap L^p(\Omega)$.

Hint: Find u_k smooth and bounded in L^p such that $Tu_k \to \infty$

18. ([Fr] p.11, #1) Let Ω be a bounded domain with $\partial\Omega$ in C^m and let $\phi \in C^m(\partial\Omega)$. Prove that there exists a function $\Phi \in C^m(\mathbb{R}^n)$ such that $\Phi = \phi$ on $\partial\Omega$.

Hint: Use partition of unit.

19. Let $H = \{x_n > 0\}$ (upper half plane) and $u \in C_0^\infty(\bar{H})$. Prove that

$$\int_{\partial H} |u|^p \leq C \int_H |u|^p + |u_{x_n}|^p$$

Hint: Use Green's theorem on a large ball containing the support of u and Young's inequality

20. Let $H = \{x_n > 0\}$ (upper half plane) and $u \in C_0^\infty(\bar{H})$. Let $v = u$ in $\bar{H}$ and $v(\cdot, x_n) =$

$-3u(\cdot, -x_n) + 4u(\cdot, -x_n/2)$ in $\bar{H}^c$ (high order reflection). Prove that $v \in C^1(\bar{H})$.

Hint: look at v_{x_n} on ∂H

21. (Adaptaded from [Fr] p.10) Suppose E_0 is a open half ball contained in B_0 an open ball with center x_0 in $y_n = 0$. Suppose $v \in C^m(\bar{E}_0)$. Let us extend v for for $y_n < 0$:

$$v(\cdot, y_n) = \sum_{i=1}^{m+1} c_j v(\cdot, -y_n/j),$$

where the c_j satisfy

$$\sum_{i=1}^{m+1} c_j(-1/j)^k = 1$$

for $0 \le k \le m$. Prove that this yields an extension of v (high order reflection) such that $v \in C^m(B_0)$.

1.3 Imbeddings and Interpolation

22. ([Ev] Sec 2.5, #3) Assume $0 < \beta < \gamma \le 1$. Let θ be defined by $\gamma = \theta \cdot 1 + (1-\theta)\cdot\beta$ and $\|\cdot\|_\eta$ be the η-Hölder norm. Prove the interpolation inequality

$$\|u\|_\gamma \le C\|u\|_1^\theta \|u\|_\beta^{1-\theta}.$$

23. ([Jo] p.169, #3) Prove that $H^1(0,2) \subset L^\infty(0,2)$, more precisely

$$g^2(x) \le 2\int_0^2 (g^2(y) + g'^2(y))dy$$

for all $x \in (0,2)$.

Hint: $g^2(x) = \int_x^{x+1}(g(y) - \int_x^y g'(z)dx)^2 dy$ and $(a-b)^2 \le 2a^2 + 2b^2$; assume first $0 < x < 1$ then use symmetry.

24. ([Fr] p.149, #1) If $u \in H^{m+1}(a,b)$, then u can be identified with a function in $C^m(a,b)$.

25. ([Ta] p.275, #6) Show that $H^k(\mathbb{R})$ is an algebra for $k > n/2$, i.e., $u, v \in H^k(\mathbb{R})$ implies that $uv \in H^k(\mathbb{R})$.

26. ([Ta] p.157, #1) Prove the estimate

$$(1-\varepsilon)\|u\|^2 \le |u(0)|^2 + C_\varepsilon \|u'\|^2$$

where $\|\cdot\|$ is the $L^2(0,1)$ norm. What is the best value of C_ε that will work?

27. ([GT] p.30, #2.15) Let $u \in C^2(\overline{\Omega}), u = 0$ on $\partial\Omega$ (smooth). Prove the interpolation inequality: For every $\varepsilon > 0$,

$$\int_\Omega |\nabla u|^2 \le \varepsilon \int_\Omega (\Delta u)^2 + \frac{1}{4\varepsilon}\int_\Omega u^2.$$

28. ([Ev] Sec 2.5, #8) Integrate by parts to prove the interpolation inequality:

$$\|Du\|^2 \le C\|u\|\|D^2u\|$$

for all u smooth with compact support. By approximation, prove the inequality if $u \in H^2 \cap H_0^1$, where $\|\cdot\|$ is the L^2 norm.

29. ([Ev] Sec 2.5, #18, [Jo] p. 169, [Ta] p.272)

(a) Prove using Fourier transform that if $s > n/2$, then $u \in H^s(\mathbb{R}^n)$ is bounded and continuous with bound

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)},$$

with C depending only on s and n.

Hint: It suffices to prove that $\widehat{u} \in L^1(\mathbb{R}^n)$. Note that $\int(1+|\xi|^{-2s})d\xi < \infty$

(b) If $s > n/2 + k$, then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

Hint: Induction and (a).

30. ([Io] p.356, #20, [Ta] p. 273) If $f \in H^s(\mathbb{R}^n)$, $s > n/2$, then there exists $\alpha \in (0,1)$ such that

$$|f(x+h) - f(x)| \le C|h|^\alpha \|f\|_{H^s}$$

for all $x, h \in \mathbb{R}^n$, with C defined by

$$2^{1-\alpha}(2\pi)^{-n/2}\left[\int_{\mathbb{R}^n}(1+|\xi|^2)^{-(s-\alpha)}\,d\xi\right]^{1/2}.$$

Therefore $H^s(\mathbb{R}^n) \subset C^{0,\alpha}(\mathbb{R}^n)$ (space of Hölder continuous functions of order α).

Hint: Use Fourier transform and prove that $|e^{iz} - e^{i\xi}| \le 2^{1-\alpha}|z-\xi|^\alpha$ for all $\alpha \in [0,1]$, $z, \xi \in \mathbb{C}$ using: If $a, b, c \ge 0$ are such that $a \le \min(b,c)$, then $a \le b^{1-\alpha}c^\alpha$ for all $\alpha \in [0,1]$.

31. ([IU] Fall 1995) Compute the Fourier transform of the characteristic function T of the unit square in $\mathbb{R}^2$ and show that $T \in H^s$ for $s < 1/2$.

1.4 Compactness

32. ([Fr] p.30, #3) Prove that the imbedding from $C^{m,\alpha}(\Omega)$ (Hölder space) into $C^{m,\beta}(\Omega)$, where $0 < \beta < \alpha < 1$ is a compact imbedding.

Hint: Prove first for $m = 0$.

33. ([IU] Winter 1992) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary and let $f \in C_0^\infty(\Omega)$. Suppose $u_k \in C_0^\infty(\Omega)$ satisfies $\Delta u_k = f$ and has the property that $\|u_k\|_{L^2(\Omega)} \le M$ for some positive constant M. Prove that exist a function $u \in C^\infty(\Omega)$ satisfying $\Delta u = f$ and a subsequence k_j such that u_{k_j} converges uniformly to u on each compact subset of Ω.

34. ([IU] Winter 1992)

Assume $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary and let $f \in C_0^\infty(\Omega)$. Suppose $u_k \in C^\infty(\Omega)$ satisfies $\Delta u_k + u_k = f$ and has the property that for some positive number M, $\|u_k\|_{L^2(\Omega)} \leq M$ for $k = 1, 2, \ldots,$. Prove that there exist a function $u \in C^\infty(\Omega)$ satisfying $\Delta u + u = f$ and a subsequence u_{k_j} which converges uniformly to u on each compact subset of Ω.

35. ([IU] Fall 1993) Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded set with smooth boundary. Consider a sequence of vector-valued functions $f_k : \Omega \to \mathbb{R}^n$ satisfying the uniform bound $\|f_k\|_{L^2} \leq M$ for some positive constant M.

(a) for each k, prove the existence of a weak solution u_k to the problem $\Delta u_k = \text{div} f_k$ in Ω and $u_k = 0$ on $\partial\Omega$.

(b) show that there exists a subsequence k_j such that f_{k_j} converges weakly to f in L^2 and u_{k_j} converges weakly to u in H_0^1 where u solves $\Delta u = \text{div} f$ in Ω and $u = 0$ on $\partial\Omega$.

2 Fredholm Alternative

1. ([Ga] p.361, #1) Solve the Fredholm integral equation below explicitly:

$$\phi(s) - \lambda \int_{-1}^{1} st\phi(t)\, dt = f(s).$$

Hint: $\phi = f + \lambda K\phi$, $\phi = f + \lambda Kf + \lambda^2 K^2 f + \cdots$

2. ([Ga] p.361, #2) Show that the Volterra integral equation

$$\phi(s) - \lambda \int_{0}^{s} K(s,t)\phi(t)\, dt = f(s).$$

can be solved by the method of successive approximations, regardless of the value of the parameter λ, provided the kernel $K(s,t)$ is bounded.

Hint: Geometric series and last exercise.

3. ([IU] Fall 1980)

(a) Find the eigenvalues and eigenfunctions of the operator $K : L^2(0,1) \to L^2(0,1)$ defined by

$$Ku(x) = \int_0^1 (x+y)u(y)\, dy$$

(b) Hence describe necessary and sufficient conditions for the existence of a solution of the equation $u(x) = x + \lambda Ku(x)$.

4. ([IU] Fall 1981) Discuss completely existence and uniqueness of L^2-solutions to the equation

$$\phi(x) = f(x) + \lambda \int_0^1 (xy + \sqrt{xy})\phi(y)\, dy$$

for all values $\lambda \in \mathbb{C}$. Assume $f \in L^2(0,1)$.

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