Exercises in PDE – Part II
Version V0.40 May 2007
Marco Aurélio P. Cabral
Institute of Matematics – UFRJ

1 Distributions and Fundamental solutions

1.1 Distributions

1. Let $f(x) = x^2 + 1$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Compute the derivative of f in the sense of distributions.

2. ([Jo] p.92, #6) Suppose $f \in (\mathbb{R})$. Define

$$f_h(x) = \frac{f(x+h) - f(x)}{h}.$$

Show that $\lim_{h \to 0} f_h = f'$ in the sense of distributions.

3.

(a) Let T be a distribution in $\mathbb{R}^2$. Prove that $D_x D_y T = D_y D_x T$. This justifies the multi-index notation for distributions since $D_{xy}T = D_{yx}T$.

(b) More generally, given any distribution T and multi-indices α, β, $D^{\alpha+\beta}T = D^\alpha D^\beta T = D^\beta D^\alpha T$.

4. ([Ra] p.257, #8, 9, 10) Let χ be the characteristic function of the first quadrant on $\mathbb{R}^2$, $\{(x, y);\ x > 0, y > 0\}$. Compute

(a) $\partial_x \chi$ and $\partial_y \chi$.

(b) $\partial_x \partial_y \chi$.

(c) repeat this for χ the characteristic function of $\{(x, y);\ xy > 0\}$.

(d) repeat this for $f(x, y) = xy/(x^2 + y^2)$ for $x, y \neq 0$.

Hint: away from 0 use calculus.

5. ([Ra] p.257, #2) Let $u(x) = |\sin(x)|$. Compute $u^{(n)}$ is the distribution sense.

6. ([Io] p.210, Theorem] 5.2) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, C^1 by parts function with a finite number of discontinuity points located at $x_1, \ldots, x_k$. Denote df/dx the derivative in the usual sense of f and suppose that it is bounded. Then, in the sense of distributions,

$$f' = \frac{df}{dx} + \sum_{j=1}^{k} [f(x_j+) - f(x_j-)]\delta_{x_j}.$$

Obs: $f(x+)$ and $f(x-)$ are the lateral limits (left and right) at the point $x \in \mathbb{R}$.

7. ([Jo] p.93, #6) Let $f_k(x)$ be a sequence of continuous non-negative functions defined in $\mathbb{R}^n$ such that

$$\int f_k(x)\, dx = 1; \quad f_k(x) = 0 \text{ for } |x - \xi| > \frac{1}{k}.$$

Show that $f_k \to \delta_\xi$ (in the sense of distributions) when $k \to \infty$.

8. Let $\varphi \in L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 1.$$

Define $\varphi_s(x) = s^{-n}\varphi(x/s)$ for $0 < s \leq 1$.

(a) ([Io] p.353, #4; [Ra] p.250; [Ev] Appendix C.4) Suppose $\varphi(x) \geq 0$. Show that $\varphi_s \to \delta_0$ (in the distribution sense) when $s \to 0^+$.

(b) ([IU] Fall 1988) Suppose $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let $f \in L^1(\mathbb{R}^n)$ be a continuous function such that $\lim_{x \to \infty} f(x) = 0$. Determine whether the convolution $\varphi_s * f$ converges to f uniformly as $s \to 0^+$.

9. ([Ra] p.257, #4) Find the most general solution $T \in \mathcal{D}'(\mathbb{R})$ of the following equations:

(a) $xT = 0$; (b) $xT' = 0$

(c) $x^2T = \delta$; (d) $T' = \delta$.

10. ([IU] Winter 1983) Let T be a distribution such that $T' = 0$ on $\mathbb{R}$. Show that T is a constant.

Hint: choose α smooth with compact support such that $\int_{\mathbb{R}} \alpha = 1$. Note that any $\phi \in \mathcal{D}(\mathbb{R})$ can be written in the form $\phi = g + (\int_{\mathbb{R}} \phi)\alpha$ with $g \in \mathcal{D}(\mathbb{R})$ and $\int_{\mathbb{R}} g = 0$.

11. The function $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$, for $(x, y) \neq 0$ and $f(0, 0) = 0$ is a well known example from calculus of a function such that $D_{xy}f \neq D_{yx}f$. Compute $D_{xy}f$ in the distribution sense.

Hint: $D_{xy}f(0, 0) = -1$ and $D_{yx}f(0, 0) = 1$.

12.

(a) ([F] p.190, #11.2) Let $\varphi(x) = e^{1/(1-x^2)}$ for $|x| \leq 1$, $\varphi(x) = 0$ otherwise. Show that $\varphi \in \mathcal{D}(\mathbb{R})$ (C^∞ with compact support).

(b) ([F] p.190, #11.3) use (a) to define $\rho : \mathbb{R}^n \to \mathbb{R}$ in $C^\infty(\mathbb{R}^n)$ and support in $|x| \leq 1$ with $\rho(x) \geq 0$ and $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$;

(c) Let $\Omega \subset \mathbb{R}^n$ be an open set. Prove that $\mathcal{D}(\Omega) \neq 0$.

13. ([Th] p.151, #1) If $f \in L^1_{\text{loc}}(\Omega)$ and $T_f = 0$, then $f = 0$ a.e. (by definition $T_f(\psi) = \int f\psi$).

14. ([Io] p.355, #12) (a) Prove that $\delta \notin L^1_{\text{loc}}(\mathbb{R}^n)$.

(b) Prove that δ' is not a measure.

15. ([Th] p.151, #2) Prove that if $u \in \mathcal{D}'(\Omega)$ and the support of u has zero measure, then there is no $f \in L^1_{\text{loc}}(\Omega)$ such that $T_f = u$. (generalization of (a) from last exercise).

16. ([Ra] p.249) Prove that a linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is a distribution if and only if for every compact $K \subset \Omega$ there is an integer n (depending on K and T) and $C \in \mathbb{R}$ such that for all $\varphi \in \mathcal{D}(\Omega)$ with support in K

$$|T \cdot \varphi| \leq C\|\varphi\|_{C^n}.$$

Obs: If n does not depend on K we say that T has order n. Note that c may depend on K. This set is the dual of $C_0^n(\Omega)$, denoted by $C^{-n}(\Omega)$.
Hint: the "if" part is easy; the "only if" can be proved by contradiction using a sequence φ_n such that $|T \cdot \varphi_n| > 1$ and $\|\varphi_n\|_{C^n} < 1/n$.

17. (a) ([Ra] p.250) prove that the distribution T defined below is a distribution of infinite order in $(0,1)$: $T \cdot \phi = \sum_{k=0}^{\infty} \phi^{(k)}(1/k)$.

(b) ([Ra] p.258, #15) prove that the distribution T defined below is a distribution of order 1 in $\mathbb{R}$:
$T \cdot \phi = \sum_{n=1}^{\infty} 1/n(\phi(1/n) - \phi(0))$.
Hint: Use that $\sum(1/n)^2 < \infty$.

18. Define $T \cdot \phi = \sum_{k=0}^{\infty} \phi^{(k)}(k)$.

(a) Prove that T defines a distribution in $\mathbb{R}$.

(b) Prove that there is no f continuous and $j \in \mathbb{N}$ such that $T = f^{(j)}$ (j-th derivative). ([Jo] p.92, #5)

(c) Confront (b) with the following theorem : (see [RS] chap. V) Let $f \in \mathcal{S}'(\mathbb{R}^n)$ (tempered distribution), then there exist g continuous, polynomial bounded, i.e.,

$$|g(x)| \leq C(1 + |x|^2)^k$$

for some $C > 0$, $k \in \mathbb{N}$ and $|x|$ sufficiently large, and a multi-índex β such that $f = \partial^\beta g$. ([Io] p.333)

19. Prove that the only distributions on $\mathbb{R}^n$ with support equal to the single point $\{0\}$ are finite linear combinations of the derivatives of δ.

20. ([IU] Fall 1992) Let $f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, and let $\alpha \in C^\infty(\mathbb{R}^n)$ have at most polynomial growth at infinity. Prove that $\alpha f \in \mathcal{S}'$ (tempered distributions).

21.

(a) Let $f \in \mathcal{S}(\mathbb{R})$ (space of Schwartz of C^∞, rapidly decaying functions) such that $f(0) = 0$. Prove that $f(x) = xg(x)$, for $x \in \mathbb{R}$ and $g \in \mathcal{S}(\mathbb{R})$.
Obs: This exercise generalizes the idea that a polynomial with a root can be factored. ([Io] p.222, #6)

(b) Let $f \in \mathcal{S}(\mathbb{R}^n)$ (space of Schwartz of C^∞ rapidly decaying functions) such that $f(0) = 0$. Prove that

$$f(x) = \sum_{j=1}^{n} x_j g_j(x),$$

for $x \in \mathbb{R}^n$ where $g_j \in \mathcal{S}(\mathbb{R}^n)$.
Hint: $f(x) = \int_0^1 \frac{d}{dt} f(tx)\, dt$. ([Io] p.354, #7)

22. ([Ra] p.257, #5) Let T be the distribution on $\mathbb{R}^n$ defined by

$$T \cdot \varphi = \int_{|z|=1} \varphi$$

(a) compute $\partial_{x_i} T$;
Hint: Use Green's theorem.

(b) compute ΔT.

23. ([IU] Fall 1982)

(a) Prove that the distribution T (known as Cauchy principal value) defined below is a distribution of order 1 in $\mathbb{R}$:

$$T \cdot \varphi = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x}\, dx.$$

(b) Let $u(x) = \log(x)$ for $x > 0$ and $u(x) = 0$ for $x \leq 0$. Compute u' (in the distribution sense) and relate in terms of (a).
Answer: $u' \cdot \varphi =$

$$\int_0^1 (\varphi(x) - \varphi(0))/x\, dx + \int_1^\infty \varphi(x)/x\, dx.$$

(c) Compute the derivative of $\log(|x|) \in L^1_{\text{loc}}(\mathbb{R})$.

24. ([Z] p.182 #2.9 (c)) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function that has zeros only in $x_1, \ldots, x_N$. Suppose that $f'(x_j) \neq 0$ for all j. Give sense to the identity

$$\delta(f(x)) = \sum_{j=1}^{N} \delta_{x_j} |f'(x_j)|^{-1}.$$

Hint: Change of variables. One consequence used often in Physics is

$$\delta(x^2 - a^2) = \frac{\delta_a + \delta_{-a}}{2|a|}.$$

25. Various extensions are possible for distributions (see references below in R. E. Edwards "Functional Analysis"; Dover 1995):

(a) replace $\mathbb{R}^n$ by a real differentiable manifold of class C^∞ (Schwartz [1] and [2] and de Rham [1]);

(b) distributions acting on differential forms of arbitrary degree rather then on functions (zero degree differential forms): the so-called currents of de Rham (de Rham [1]);

(c) replace $\mathbb{R}^n$ by a locally compact abelian group (Riss[1]).

1.2 Fundamental Solutions

From [Wi] : In mathematics, a fundamental solution for a linear partial differential operator L is a formulation in the language of distribution theory of the older idea of a Green's function. In terms of the Dirac delta function δ, a fundamental solution f is the solution of the inhomogeneous equation $Lu = \delta$.

Here f is a priori only assumed to be a Schwartz distribution.

This concept was long known for the Laplacian in two and three dimensions. It was investigated for all dimensions for the Laplacian by Marcel Riesz. The existence of a fundamental solution for any operator with constant coefficients – the most important case, directly linked to the possibility of using convolution to solve an arbitrary RHS – was shown by Malgrange and Ehrenpreis.

The motivation to find the fundamental solution is because once one finds the fundamental solution, it is easy to find the desired solution of the original equation. In fact, this process is achieved by convolution.

Fundamental solutions also play an important role in the numerical solution of partial differential equations by the boundary element method.

Notice: The definition of fundamental solutions for systems usually is not covered in literature.

Definition: Let δ be the Dirac delta operator concentrated at the origin. We write $\delta(x-\xi)$ for the Dirac operator concentrated at ξ. Let L be any linear differential operator. We say that Φ is a *fundamental solution* with pole ξ if $L\Phi = \delta_\xi$ in the sense of distributions.

With the fundamental solution Φ with pole $\xi = 0$ one can solve $Lv = f$ for any (smooth) f: $v = \Phi * f$ (convolution of Φ and f).

This is true since $L(\Phi * f) = L(\Phi) * f = \delta * f = f$.

26. ([Jo] p.92) (a) Show that $u(x) = 1/2|x-\xi|$ is a fundamental solution with pole ξ of the operator $L = d^2/dx^2$;

(b) Use it to solve $u'' = f$.

Hint: $H' = \delta$ (H is the Heaviside function).

27. ([Jo] p.92, #2) Show that

$$u(x,y) = \begin{cases} 1; & \text{for } x > \xi_x, y > \xi_y \\ 0; & \text{otherwise.} \end{cases}$$

is a fundamental solution with pole (ξ_x, ξ_y) of the operator $L = \partial^2/\partial x \partial y$ in the xy-plane.

28. ([Z] p.179 #2.9 (a)) Show that

$$u(x) = \begin{cases} \dfrac{x^{n-1}}{(n-1)!}; & x \geq 0, \\ 0; & x < 0, \end{cases}$$

is a fundamental solution with pole 0 of the operator $L = d^n/dx^n$.

$$u^{(n)} = \delta_0.$$

29. ([Z] p.413, #5.1) Show that $u(x) = 1/(2\pi)\exp(i\lambda|x|)$ is a fundamental solution with pole 0 of the operator (reduced wave operator) $L = d^2/dx^2 + \lambda^2$.

Answer: $u(x) = \cos(\lambda|x|)/(4\pi|x|)$.

30. ([IU] Fall 1990) Find all solutions $u \in \mathcal{D}'(\mathbb{R})$ of the differential equation $u'' + u = \delta$.

31. ([Co]) Show that $u(x) = \exp(-m|x|)/(2m)$ is a fundamental solution with pole 0 of the Helmholtz 1-D operator $L = -d^2/dx^2 + m^2$ with $m > 0$.

1.3 Applications to PDE

1.3.1 Laplace equation

32. ([IU] Fall 1988) Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\Delta u \in C_0^k(\mathbb{R}^n)$ for some $0 \leq k \leq \infty$. Show that $u \in C^{k+1}(\mathbb{R}^n)$.

33. ([IU] Fall 1989) Let $f \in C^\infty(\mathbb{R}^n)$ and suppose T is a distribution that satisfies $\Delta T = f$. Prove that $T \in C^\infty(\mathbb{R}^n)$.

Hint: Use the fact that if S is a distribution satisfying $\Delta S = 0$, then S is an harmonic function.

34. Prove that Φ, the fundamental solution of Laplace's equation, satisfies $\Delta\Phi = \delta_0$.

Hint: use theory of Laplace's equation: it is not easy!

35. ([IU] Fall 1991) Compute a solution u of $\Delta u = S$ in $\mathbb{R}^2$ and $S \in \mathcal{D}'(\mathbb{R}^2)$ is defined by

$$S \cdot \phi = \int_0^1 \phi_y(x,0)\,dx$$

for $\phi \in \mathcal{D}(\mathbb{R}^2)$. Express u in closed form without integrals.

Hint: $u = \Phi * S$

1.3.2 Transport and Wave equations

36. ([Jo] p.92, #1) Show that for a continuous function f the expression $u = f(x - ct)$ is a solution in the sense of distributions of the (transport) equation $u_t + cu_x = 0$.

Hint: Change coordinates $y_1 = x - ct, y_2 = x$. Take as test function $\psi(y_1)X(y_2)$.

37. (1D wave eq)

(a) ([Jo] p.92, #3 and [IU] Fall 1980) Let H be the characteristic function of $[0,\infty]$, i.e., $H(x)=1$ if $x\geq 0$ and $H(x)=0$ otherwise. Prove that

$$u(x,t)=1/2H(t-|x|)$$

is a fundamental solution of the 1D wave operator $L=\partial^2/\partial t^2-\partial^2/\partial x^2$ in the xt-plane.

(b) Use it to show that a solution of the 1D wave equation $u_{tt}-u_{xx}=f$ is given by

$$u(x,t)=\frac{1}{2}\int_0^t\int_{x-s}^{x+s}f(y,t-s)\,dy\,ds.$$

Note that if we define Ω as the triangle on $\mathbb{R}^2$ with base on the x-axis given by $[x-t,x+t]$ and vertices (x,t) then

$$u(x,t)=\frac{1}{2}\int_\Omega f(y,s)\,dy\,ds.$$

38. (3D wave equation) Let T be a distribution defined by

$$T\cdot\varphi=\frac{1}{4\pi}\int_0^\infty t^{-1}\left(\int_{|x|=t}\varphi(x,t)\,dS_x\right)\,dt.$$

(a) Show that

$$T=\frac{1}{4\pi}\frac{\delta(t-|x|)}{|x|}=\frac{1}{4\pi}\frac{\delta(t=|x|)}{t},$$

a spherical wave which emanates from the origin and propagates at speed 1. The amplitude of the wave is inversely proportional to the distance from the origin.

(b) ([Z] p.182 #2.9 (c)) Show that T is a fundamental solution of the 3D wave equation

$$T_{tt}-\Delta T=\delta_0.$$

(c) Use it to show that a solution of the 3D wave equation $u_{tt}-\Delta u=f$ is given by

$$u(x,t)=\frac{1}{4\pi}\int_{B(x,t)}\frac{f(y,t-|y-x|)}{|y-x|}\,dy.$$

1.3.3 Heat Equations

39. (3D heat equation)

(a) ([Z] p.182 #2.9 (c)) Show that

$$\Phi(x,t)=\begin{cases}\dfrac{1}{(4\pi t)^{3/2}}e^{-|x|^2/(4t)}; & t>0,\\ 0; & t\leq 0,\end{cases}$$

is a fundamental solution of the 3D heat equation

$$\Phi_t-\Delta\Phi=\delta_0.$$

(b) Use it to show that a solution $u(x,t)$ of $u_t-\Delta u=f$ is given by

$$\int_0^t\frac{1}{(4\pi(t-s))^{3/2}}\int_{\mathbb{R}^3}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)\,dy\,ds$$

40. Let $K_t(x)=(4\pi t)^{-n/2}\exp(-|x|^2(4t)^{-1})$ (heat kernel). Prove that $K_t\to\delta$ in the distribution sense when $t\to 0+$

Hint: Use the theory of the heat equation.

1.3.4 Other Equations

41. ([IU] Fall 1995) Let T be a temperate distribution on $\mathbb{R}^2$ satisfying $\Delta T+D_xT+D_yT+T=0$.

(a) show that $T\in C^\infty(\mathbb{R}^2)$;

(b) if T has a limit as $\|(x,y)\|\to\infty$, prove that $T\equiv 0$.

42. ([IU] Winter 1992) The curl operator in two dimensions acting on a vector field $U(x,y)=(u(x,y),v(x,y))$ is defined as $\nabla\times U=u_y-v_x$.

(a) give a definition of what it means for a n (not necessarily continuous) vector field U to have curl zero in the sense of distributions.

(b) Let Γ be a smooth curve dividing $\mathbb{R}^2$ into two disjoint open regions Ω_1 and Ω_2. Suppose U takes the form $U\equiv U_1$ on Ω_1 and $U\equiv U_2$ on Ω_2, with U_1 and U_2 are (different) constants vectors. If U has curl zero in the sense of distributions, describe as completely as possible the curve Γ.

43. ([IU] Fall 1993) Let Σ be a smooth hypersurface dividing $\mathbb{R}^n$ into two disjoint open regions Ω_1 and Ω_2, Denote $\nu=\nu(x)$ the unit normal to Σ pointing into Ω_2. Suppose $v:\mathbb{R}^n\to\mathbb{R}^n$ solves the equation $\mathrm{div}v(x)=b(x)$ for $x\in\mathbb{R}^n-\Sigma$ in the classical sense, where b is a continuous function. Assume furthermore that $v\in C(\bar\Omega_1)\cap C(\bar\Omega_2)$ (but not necessarily in $C(\mathbb{R}^n)$!) with

$$v^+(x_0)=\lim_{x\to x_0;x\in\Omega_1}v(x)$$

and

$$v^-(x_0)=\lim_{x\to x_0;x\in\Omega_2}v(x).$$

for each $x_0\in\Sigma$. Derive a necessary and sufficient condition in terms of v^+,v^- and ν for v to be a distribution solution on all $\mathbb{R}^n$.

2 Fourier Transform

2.1 General Theory

1. ([Th] p.168, #1) Compute the Fourier transform of the following functions:

(a) $f(t)=e^{-|t|\lambda}$, para $\lambda>0$.

(b) $f(t) = P(t)e^{-t^2/2}$, onde $P(t)$ é um polinômio.

(c) $f(t) = \chi_{[a,b]}(t)$, função característica do intervalo $[a,b]$.

(d) $f(t) = e^{-t\lambda}$, para $\lambda > 0$, $t \geq 0$, $f(t) = 0$ para $t < 0$.

2. ([Ha] p.346, #9.4.6) Compute the Fourier transform of the function

$$f(x) = \begin{cases} 0; & |x| > a \\ 1; & |x| < a. \end{cases}$$

Answer: $\sin(a\xi)/(\pi\xi)$.

3. ([Io] p.353, #2) Let $f \in L^1(\mathbb{R}^n)$. Define,

$$(\tau_a f)(x) = f(x-a)$$

$$(E_a f)(x) = e^{ia\cdot x} f(x),$$

$$h_\lambda(x) = f(x/\lambda),$$

with $x, a \in \mathbb{R}^n, \lambda \in (0,\infty)$. Prove that

(a) $(\tau_a \widehat{f})(\xi) = (E_a f)\widehat{\ }(\xi)$.

(b) $(E_a \widehat{f})(\xi) = (\tau_{-a} f)\widehat{\ }(\xi)$.

(c) $\widehat{h}_\lambda(\xi) = \lambda^n \widehat{f}(\lambda\xi)$.

(d) Show that (c) implies that functions with a large support have Fourier transform with a sudden peak near 0 and vie-versa. Observe that the transform of Dirac's delta is a constant, it contains every frequency! ([Ha] p.347, #9.4.11 (c))

4. ([Ha] p.346, #9.4.3) Let $F(\xi)$ be the Fourier transform of $f(x)$. Prove that if $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, then $\bar{F}(\xi) = F(-\xi)$. ($\bar{z}$ is the complex conjugate of z).

5. Define $\mathcal{F}f = \widehat{f}$.

(a) Prove that $\mathcal{F}^4 = I$ (identity). ([Io] p.314)

Hint: Show directly (change of variables) that $\mathcal{F}^2 = \mathcal{F}^{-2}$. Another proof is to use (a').

(a') Let $Jh(x) = h(-x)$. Show that $\mathcal{F}^2 = J$;

(a") Show that $f^\vee(x) = \widehat{f}(-x)$;

(b) Let $\gamma(x) = \exp(-|x|^2/2)$ for $x \in \mathbb{R}^n$. Prove that $\mathcal{F}\gamma = \gamma$. ([Io] p.184, p. 194).

Hint: proof is NOT trivial: see next exercise.

(c) Use (a) to find all eigenvalues of $\mathcal{F}$.

Answer: eigenvalues $(-i)^n$ with eigenfunctions $e^{-x^2/2}H_n(x)$, where H_n are the Hermite polynomials

6. ([IU] Winter 1983) Let $\gamma(x) = \exp(-x^2/2)$ for $x \in \mathbb{R}$. Show that:

(a) $d/dx\widehat{\gamma}(\xi) = -\widehat{\xi(\xi)}$.

(b) $d/dx(\widehat{\gamma}/\gamma) = 0$

(c) $\widehat{\gamma} = \gamma$.

7. ([Th] p.168,#2) Prove that $L^2(\mathbb{R}^n)$ has an orthogonal decomposition

$$L^2(\mathbb{R}^n) = H_1 \oplus H_{-1} \oplus H_i \oplus H_{-i}$$

such that $\widehat{\phi} = \lambda\phi$ for $\phi \in H_\lambda$.

8. (Inversion formula: Proof I) (Adaptaded from [Io] p.183) Let $\widehat{g} = \mathcal{F}g$. Consider $h_a = \widehat{g}\chi_{(-a,a)}$.

(a) Prove that $\lim_{a\to\infty} h_a = \widehat{g}$.

(b) Determine $\lim_{a\to\infty} h_a^\vee$.

(c) Conclude that $(\widehat{g})^\vee = g$.

Hint: $\int_{-\infty}^{+\infty} \sin(x)/x\, dx = \pi$ as an improper Riemann integral, even though $\sin(x)/x \notin L^1$ (why ?). This is a canonical example of a Riemann integrable function that is not Lebesgue integrable: it is possible only because the support of the function is not compact.

9. (Inversion formula: Proof II)

(a) Determine $\widehat{\chi}_{(-a,a)}$.

(b) Prove directly (computing the integral) that $\widehat{\chi}^\vee = \chi$.

Hint: $\cos(k\xi)/\xi$ is odd, therefore its integral in $\mathbb{R}$ is zero. Use hint from last exercise to determine $\sin(k\xi)/\xi$: be careful when $k > 0$ and $k < 0$.

(c) Use a density of stair functions in $L^1(\mathbb{R})$ to finish the proof.

10. ([Ta] p.202, #1) (Riemmann-Lebesgue Lema) Show that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$. Show also that $\widehat{f} \in C(\mathbb{R}^n)$.

Hint: Use that $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and integrate by parts.

2.2 Distributions and FT

11. ([IU] Winter 1993) Let $P(D)$ be a constant coefficient partial differential operator on $\mathbb{R}^n$. Suppose $P(i\xi) \neq 0$ for all $\xi \neq 0$. If $u \in \mathcal{S}'$ (tempered distributions) satisfies $P(D)u = 0$. Prove that u must be a polynomial.

12. Let p be a polynomial in n variables, and assume that the set $\{x; p(ix) = 0\}$ is bounded.

(a) ([IU] Fall 1991) Prove that there exists a tempered distribution E on $\mathbb{R}^n$ such that $p(D)E - \delta \in \mathcal{S}$, the class of rapidly decreasing functions (Schwartz space).

(b) ([IU] Fall 1994) Prove that if T is a tempered distribution on $\mathbb{R}^n$ then there is a tempered distribution E such that $p(D)E - T \in C^\infty(\mathbb{R}^n)$.

13. ([IU] Winter 1992) The Fourier transform of a temperate distribution T is defined by $\widehat{T}\cdot\varphi = T\cdot\widehat{\varphi}$ for all test functions φ. Assume μ is a measure on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) < \infty$.

(a) prove that μ is a temperate distribution;

(b) Prove that $\widehat{\mu}$ is (a function) given by

$$\widehat{\mu}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\cdot\xi}\, d\mu(x).$$

14. ([Io] p.355, #17) A Dirac δ distribution over the sphere of radius $a > 0$ (used in problems of quantum mechanics) is define, for $\varphi \in \mathcal{D}(\mathbb{R})$

$$\delta_a \cdot \varphi = a^{n-1} \int_{S^{n-1}} \varphi(aw)\, dw.$$

(a) Prove that δ_a is a tempered distribution.
(b) Prove that its Fourier transform is

$$\widehat{\delta_a}(\xi) = a^{n-1}(2\pi)^{-n/2} \int_{S^{n-1}} \exp(-iaw \cdot \xi)\, dw.$$

(c) Compute (b) explicitly for $n = 3$.
Hint: The integral is invariant under rotations. Choose cylindrical coordinates with the z axis parallel to ξ.
(d) Do the same for $n \geq 2$.

15. ([Io] p.360, #32) Let T (known as Cauchy principal value) and S be defined by

$$T \cdot \varphi = \lim_{\varepsilon \to 0+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx$$

$$S \cdot \varphi = \lim_{\varepsilon \to 0+} \int_{-\infty}^{+\infty} \frac{\varphi(x)}{x + i\varepsilon}\, dx$$

(a) Prove that T and S are tempered distributions.
(b) What is the relation between them? (Is it $T = S$?)
(c) Compute the Fourier transform of these distributions.

16. (determining $\widehat{\delta}$: Proof I)
Let $\chi_{(-a,a)}$ be the characteristic function of the interval $(-a, a)$.
(a) Determine $\widehat{\chi}_{(-a,a)}$.
(b) Prove that, in the sense of distributions,

$$\lim_{a \to 0} \frac{1}{2a} \chi_{(-a,a)} = \delta_0.$$

(c) Determine, using (a),

$$\lim_{a \to 0} \frac{1}{2a} \widehat{\chi}_{(-a,a)}.$$

(d) Use (c) to prove that $\widehat{\delta_0}(\xi) = (2\pi)^{-1/2}$.
(e) Generalize last argument to prove that $\widehat{\delta_x}(\xi) = (2\pi)^{-1/2} \exp(i\xi x)$.

17. (determining $\widehat{\delta}$: Proof II) ([Ha] p.348, #9.4.18)
(a) Consider $g_\beta(x) = \alpha \exp(-\beta(x - x_0)^2)$. Determine α such that $\int_R g_\beta(x)\, dx = 1$.
(b) Show that $\lim_{\beta \to \infty} g_\beta = \delta_{x_0}$ (in the sense of distributions).
(c) Determine $\lim_{\beta \to \infty} \widehat{g}_\beta$.
(d) Use (c) to prove that $\widehat{\delta}_{x_0}(\xi) = (2\pi)^{-1/2} \exp(i\xi x)$.

2.3 Applications to PDE

2.3.1 Heat Equation

18.
(a) Solve for $\widehat{u}$ for $u_t - \Delta u = 0$ in $\mathbb{R}^n$ with $u(x, 0) = f(x)$.
(b) prove that the L^2 norm of u goes to zero as $t \to \infty$.

19. ([IU] Fall 1994) Let u be the bounded solution of the heat equation $u_t = \Delta u$ in $\mathbb{R}^n$, $t > 0$ and $u(x, 0) = \phi(x)$ where $\phi \in \mathcal{S}$ the class of rapidly decreasing functions (Schwartz class).
(a) Prove that there is a constant C depending on ϕ such that

$$\|u(\cdot, t)\|_{L^2} \leq C(1 + t)^{-n/4}.$$

(b) Assume in addition that $\int_{\mathbb{R}^n} \phi = 0$ and prove that

$$\|u(\cdot, t)\|_{L^2} \leq C(1 + t)^{-(n+2)/4}.$$

20. ([IU] Fall 1988) Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ solve the heat equation $u_t = \Delta u$, and satisfy the initial condition $u(x, 0) = u_0(x) \in L^2(\mathbb{R}^n)$. Given $t > 0$ derive a bound for $\|D_x^\alpha u(\cdot, t)\|_{L^2(\mathbb{R}^n)}$ in terms of t, α, and $\|u_0\|_{L^2(\mathbb{R}^n)}$.

21. ([Winter 1991] p.L)et $u \in C^2(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T])$ be a solution to the problem $u_t = \Delta u - u^3$ for $x \in \Omega$, $t > 0$, $u = 0$ in $\partial\Omega$, $u(x, 0) = f(x)$ for $x \in \Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$.
(a) Prove that $\|u\|_{L^2}(t) \leq \|f\|_{L^2}$ for all $t \in (0, T]$.
(b) Prove that $\|u\|_{L^\infty}(t) \leq \|f\|_{L^\infty}$ for all $t \in (0, T]$.
(c) Prove that the solution of this problem is unique.
(d) Suppose $f \in L^4 \cap H^1$, prove that

$$\|u\|_{L^4}^4(t) + \|u\|_{H^1}^2(t) \leq \|f\|_{L^4}^4 + \|f\|_{H^1}^2.$$

22. ([IU] Winter 1992) Let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary and suppose $u \in C^\infty(\Omega \times [0, \infty))$ is a solution of the equation $u_t - \Delta u = f$ with $u = 0$ on $\partial\Omega \times [0, \infty)$. Assume that

$$\lim_{t \to \infty} \int_\Omega f(x, t)^2\, dx = 0.$$

Let $\psi(t) = \displaystyle\int_\Omega u(x, t)^2\, dx$. Prove that $\psi(t) \to 0$ as $t \to \infty$.

2.3.2 Wave Equation

23.

(a) Solve for $\widehat{u}$ for $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n$ with $u(x,0) = 0$ and $u_t(x,0) = g(x)$.

Answer: $\widehat{u} = \widehat{g}(\xi)\sin(|\xi|t)/|\xi|$.

(b) prove that the L^2 norm of u is preserved.

(c) determine u for $\mathbb{R}$.

Hint: $\widehat{\chi}_{[-t,t]}(\xi) = \sin(|\xi|t)/(\pi|\xi|)$.

(d) Solve for $\widehat{u}$ for $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n$ with $u(x,0) = f(x,0)$ and $u_t(x,0) = g(x)$ and show that

$$\|u\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2.$$

24. ([IU] Winter 1995) (equipartion of energy principle) Let u be a solution of the wave equation in $\mathbb{R}^n$ $u_{tt} = \Delta u$ with initial data $u = g$ and $u_t = h$. Suppose that $g, h \in \mathcal{S}$, the Schwartz class of rapidly decaying functions. Define the kinetic energy by $k(t) = \int_{\mathbb{R}} u_t^2(x,t)dx$ and the potential energy as $p(t) = \int_{\mathbb{R}} (\nabla_x u)^2(x,t)dx$. Prove that

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} k(t) = \frac{1}{2}(p(0) + k(0)).$$

Hint: $2\sin^2(x) = 1 - \cos(2x)$, $2\sin(x)\cos(x) = \sin(2x)$

25. ([IU] Fall 1992) Consider the solution of the 3D wave equation $u_{tt} = \Delta u$ with initial data $u = 0$ and $u_t = h \in C_0^\infty$.

(a) Prove that there is a positive constant C such that

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} u^2(x,t)\,dx \le C.$$

Hint: $2\sin^2(x) = 1 - \cos(2x)$

(b) Prove that there is a positive constant C_1 such that

$$t \cdot \max_x |u| \ge C_1$$

for all sufficiently large t.

26. ([IU] Fall 1995) Consider the solution $u \in C^\infty$ of the wave equation in $\mathbb{R}^n$, $u_{tt} - \Delta u = 0$ for $t > 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, with $f, g \in C_0^\infty$.

(a) for $n \ge 3$, show that $\|u\|_{L^2}(t)$ has a finite limit as $t \to \infty$, and identify that limit. Show that there exists a constant C independent of n, f, g such that

$$\|u\|_{L^2}^2(t) \le \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + C\|g\|_{L^1}^2.$$

(b) for $n \le 2$, show that $\|u\|_{L^2}(t)$ has a finite limit as $t \to \infty$ if, and only if, $\int_{\mathbb{R}^n} g = 0$, in which case, there exists a constant C independent of n, f, g such that

$$\|u\|_{L^2}^2(t) \le \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + C\|xg\|_{L^1}^2.$$

27. ([IU] Fall 1993) Assume g is smooth and in $L^1(\mathbb{R}^n)$. Let $\widehat{u}(\xi,t)$ be the Fourier transform (with respect to x) of the solution u of the wave equation in $\mathbb{R}^n$, $u_{tt} - \Delta u = 0$ for $t > 0$, $u(x,0) = 0$, $u_t(x,0) = g(x)$. Show that the L^p norm of $\widehat{u}(\cdot,t)$ at time t for $p > n$ is bounded by a constant times $t^{1-n/p}$.

28. ([IU] Winter 1995) Consider the solution $u \in C^\infty$ of the inhomogeneous wave equation in $\mathbb{R}^n$, $u_{tt} - \Delta u = f$ for $t > 0$, $u(x,0) = u_t(x,0) = 0$, with $f \in \mathcal{S}$, the Schwartz class of rapidly decaying functions. Prove that

$$\|u\|_{L^2(\mathbb{R}^n\times[0,T])} \le C\|f\|_{L^2(\mathbb{R}^n\times[0,T])}.$$

29. ([Ta] p.221) Let $\widehat{R(t,\xi)} = (2\pi)^{-n/2}|\xi|^{-1}\sin(t|\xi|)$, where $R(t,x)$ is the fundamental solution of the wave equation for initial data $u(0,x) = 0, u_t(0,x) = \delta_0$. Prove that:

(a) $R(t,x) = 0$ for $|x| > |t|$ (finite propagation speed);

(b) for $n = 2$, $R(t,x) = c(t^2 - |x|^2)^{-1/2}\mathrm{sgn}(t)$ for $|x| < |t|$;

(c) for $n = 3$, $R(t,x) = (4\pi t)^{-1}\delta(|x| - |t|)$.

2.3.3 Laplace Equation

30. ([GL] p.305, #5) In 2D solve $\Delta u = 0$ for $y > 0$, $x \in \mathbb{R}$ and $u(x,0) = f(x)$ for $x \in \mathbb{R}$.

Answer: $u(x,y) = y/\pi \int f(s)\,ds/((x-s)^2+y^2)$ for $y > 0$, the classical Poisson integral formula.

31. ([Gu] p.187)

(a) Solve for $\widehat{u}$ for $-\Delta u = f$ in $\mathbb{R}^n$.

Answer: $\widehat{u} = \widehat{f}(\xi)/|\xi|^2$

(b) Show that $(1/|\xi|^2)^\vee = 1/(4\pi|x|)$ in $\mathbb{R}$.

(c) Show that $u(x) = (4\pi)^{-1}\int_{\mathbb{R}} f(y)/|x - y|\,dy$

2.3.4 Other Equations

32. ([Ha] p.358, #9.5.3)

(a) Solve the diffusion equation with convection

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$$
$$u(x,0) = f(x)$$

for $x \in \mathbb{R}$.

(b) Solve the same equation with $u(x,0) = \delta(x)$. Sketch solutions for different values of t. What is the meaning of the convection term $c\dfrac{\partial u}{\partial x}$?

33. ([Ha] p.358, #9.5.7) Solve using Fourier transform, obtaining a formula appearing $\widehat{f}$, the linearized KdV

$$u_t = ku_{xxx}; \qquad \text{for } x \in \mathbb{R}$$

$$u(x,0) = f(x).$$

34. ([IU] Winter 1991) Consider the problem $u_t = -u_{xxxx} - au_{xx}$, $u(x,0) = f(x)$, where a is a positive constant and $f \in L^1 \cap L^2$.

(a) obtain an integral representation for the solution;

(b) use it to show directly that $u \in C^\infty(\mathbb{R} \times \mathbb{R}^+)$.

3 Bibliography

[Co] Cole, Kevin D. (2007); *Internet Green's Function Library* (for Heat, Laplace and Helmoltz); `www.greensfunction.unl.edu`. Accessed in April 2007.

[Ev] Evans, Lawrence C. (1998); *Partial differential equations*; Graduate Studies in Mathematics, 19; American Mathematical Society, Providence, RI.

[Fi] Figueredo, Djairo G. (1987); *Análise de Fourier e Equações DIferenciais Parciais;* Impa.

[Fo] Folland, Gerald B. (1995); *Introduction to partial differential equations* Princeton University Press.

[Fr] Friedman, Avner (1976); *Partial differential equations*; Robert E. Krieger Publishing Co.

[Ga] Garabedian, P. R. (1986); *Partial differential equations*; Chelsea Publishing Co.

[Ge] Gerlach, U. H. (2007); *Linear mathematics in infinite dimensions; Signals, Boundary Value Problems and Special Functions*; `www.math.ohio-state-.edu /~gerlach/math/BVtypset/` Accessed in April 2007.

[GT] Gilbarg, David; Trudinger, Neil S (1998); *Elliptic partial differential equations of second order*; Springer-Verlag.

[GL] Guenther, Ronald B.; Lee, John W. (1988); *Partial differential equations of mathematical physics and integral equations*; Prentice Hall.

[Gu] Gustafson, Karl E. (1980); *Partial Differential Equations and Hilbert Spaces Methods*; John Wiley & Sons.

[Ha] Haberman, Richard (1987); *Elementary Applied Partial Differential Equations*; Prentice-Hall.

[Io] Iório, Rafael; Iório, Valéria (1988); *Equações Diferenciais Parciais: Uma Introdução*; IMPA.

[IU] Indiana University; Qualifying Exams in Applied Mathematics for a PhD in Mathematics.

[Jo] John, Fritz (1982); *Partial Differential Equations;* Springer-Verlag.

[Ra] Rauch, Jeffrey (1991); *Partial Differential Equations;* Springer-Verlag.

[RS] Reed, Michael; Simon, Barry (1975); *Methods of Modern Mathematical Physics* Vol. II, Fourier analysis, self-adjointness; Academic Press.

[Ta] Taylor, Michael E. (1996); *Partial differential equations. I. Basic theory.* Applied Mathematical Sciences, 115. Springer-Verlag.

[Th] Thayer, Javier; (1987); *Operadores Auto-Adjuntos e Equações Diferenciais Parciais*; IMPA.

[TZ] Thoe, Dale W.; Zachmanoglou, E. C. (1976); *Introduction to partial differential equations with applications*; The Williams & Wilkins Co., Baltimore, Md.

[Wi] Wikipedia (2007); `en.wikipedia.org;` Acessed in April 2007.

[Ze] Zeidler, Eberhard (1995); *Applied Funcional Analysis: Applications to mathematical physics*; Applied Mathematical Sciences, 108; Springer-Verlag.