Harmonic Functions are Real Analytic[1]

On this very short note we prove that harmonic functions are real analytic functions.

Lemma 1 is used on the inductive step of the proof of the main theorem. We prove Lemma 1 using the mean value property of harmonic functions, Green's theorem and the maximum principle.

Lemma 1 (estimate on first derivative) *Suppose w is harmonic in $\Omega \subset \mathbb{R}^n$, $\rho > 0$, $y \in \Omega$ and $B(y;\rho) \subset \Omega$. Then for any $i \in \{1, \dots, n\}$,*

$$|w_{x_i}(y)| \leq \frac{n}{\rho} \|w\|_{L^\infty(B(y;\rho))}.$$

Proof: Since w is harmonic, w_{x_i} is also harmonic. Therefore we can use the mean value property on w_{x_i}. Using also Green's theorems,

$$|w_{x_i}(y)| = \left| \frac{1}{\alpha(n)\rho^n} \int_{B(y;\rho)} w_{x_i}\, dx \right| = \left| \frac{1}{\alpha(n)\rho^n} \int_{\partial B(y;\rho)} w\nu_i\, dx \right| \leq$$

$$\leq \frac{1}{\alpha(n)\rho^n} n\alpha(n)\rho^{n-1} \|w\|_{L^\infty(\partial B(y;\rho))} = \frac{n}{\rho} \|w\|_{L^\infty(\partial B(y;\rho))}.$$

Using the maximum principle on w,

$$|w_{x_i}(y)| \leq \frac{n}{\rho} \|w\|_{L^\infty(\partial B(y;\rho))} \leq \frac{n}{\rho} \|w\|_{L^\infty(B(y;\rho))}.$$

□

Corollary 1 (Liouville's Theorem) *Any harmonic and bounded function on $\mathbb{R}^n$ is constant.*

Proof: Since is harmonic on $\mathbb{R}^n$ we can let ρ goes to infinity on Lemma 1 and conclude that its derivative is zero at any point.

□

The idea of the proof of Theorem 1 is to estimate a derivative of order k using balls with radius $r_j = j/(k+1)$, for $j = 1, \dots, k+1$. We proceed by induction on k.

Theorem 1 (estimates on higher derivatives) *Suppose u is harmonic in $\Omega \subset \mathbb{R}^n$, $r > 0$, $z \in \Omega$ and $B(z;r) \subset \Omega$. Then for any multi-index α of order k,*

$$|D^\alpha u(z)| \leq \left(\frac{kn}{r} \right)^k \|u\|_{L^\infty(B(z;r))}. \tag{1}$$

Proof: We will prove by induction on k.

For $k = 0$, $|u(z)| \leq \|u\|_{L^\infty(B(z;r))}$.

Suppose (1) is true for $|\alpha| = k$. Then given any multi-index β of order $k+1$, there exists a multi-index α of order k such that $D^\beta u = (D^\alpha u)_{x_i}$ for some $i \in \{1, \dots, n\}$. Since u is harmonic, $D^\alpha u$ is harmonic also. From Lemma 1, for $w = D^\alpha u$, $y = z$ and $\rho = r/(k+1)$, which we can apply since $B(z; r/(k+1)) \subset B(z;r) \subset \Omega$, we obtain

$$|D^\beta u(z)| \leq \frac{(k+1)n}{r} \|D^\alpha u\|_{L^\infty(B(z;r/(k+1)))}. \tag{2}$$

If $x \in B(z; r/(k+1))$, then $B(x; kr/(k+1)) \subset B(z;r) \subset \Omega$ since $r/(k+1) + kr/(k+1) = r$. Therefore, using the induction hypotheses (1) on α for $z = x$, $r = kr/(k+1)$ (we can apply it since $B(x; kr/(k+1)) \subset \Omega$)

$$|D^\alpha u(x)| \leq \left(\frac{kn(k+1)}{kr} \right)^k \|u\|_{L^\infty(B(x;kr/(k+1)))} \leq \left(\frac{n(k+1)}{r} \right)^k \|u\|_{L^\infty(B(z;r))}.$$

Using (2),

$$|D^\beta u(z)| \leq \frac{(k+1)n}{r} \left(\frac{n(k+1)}{r} \right)^k \|u\|_{L^\infty(B(z;r))} \leq \left(\frac{n(k+1)}{r} \right)^{k+1} \|u\|_{L^\infty(B(z;r))}.$$

This verify (1) for $|\beta| = k+1$.

□

Now we need an easy estimate on k^k for our main Theorem 2. This can be done using Stirling's formula (see Federer), but our proof below is more elementary.

[1] Verso 03.april.2008

Lemma 2 (estimate on k^k) *For any $k \in \mathbb{N}$,*

$$k^k < k! e^{k-1}.$$

Proof: One can prove by induction (see (J. R. Felicio; Frmula de Stirling em tempos de Maple; Revista de Matemtica Universitria, **17**, (1994)) that

$$k^k = k! \prod_{j=1}^{k-1} (1 + 1/j)^j.$$

Since $(1 + 1/j)^j < e$ (the sequence is monotone using Newton's binomial formula) for every j, $k^k < k! e^{k-1}$.

□

Theorem 2 (harmonic functions are analytic) *If u is harmonic in $\Omega \subset \mathbb{R}^n$ then u is real analytic in Ω.*

Proof: Fix any $x_0 \in \Omega$. We will show that its Taylor series is convergent in a neighborhood of x_0.

Let $r = \frac{1}{2}\text{dist}(x_0, \partial\Omega)$. For each k, the remainder $R_k(x)$ of the Taylor series of u is

$$R_k(x) = \sum_{|\alpha|=k} D^\alpha u(\xi) \frac{(x - x_0)^\alpha}{\alpha!}, \tag{3}$$

where $\xi = x_0 + t(x - x_0)$, for some $0 \leq t \leq 1$, t depending on x. Let us assume that $|x - x_0| < \rho < r$ for some $\rho > 0$ that we are going to choose later. Since $|\xi - x_0| = |t||x - x_0| \leq |x - x_0| < \rho < r$, we see that $\xi \in B(x_0; r)$. Since $B(\xi; r) \subset B(x_0; 2r) \subset \Omega$, we can apply Theorem 1 for $z = \xi$ and obtain

$$|D^\alpha u(\xi)| \leq \left(\frac{kn}{r}\right)^k \|u\|_{L^\infty(B(\xi;r))} \leq \left(\frac{kn}{r}\right)^k \|u\|_{L^\infty(B(x_0;2r))}. \tag{4}$$

Using Lemma 2 (estimate on k^k) and (4), defining $M = \|u\|_{L^\infty(B(x_0;2r))}$, we obtain

$$|D^\alpha u(\xi)| \leq M \left(\frac{kn}{r}\right)^k \leq \frac{M}{e} \left(\frac{en}{r}\right)^k k!. \tag{5}$$

It is easy to check that $|(x - x_0)^\alpha| \leq |x - x_0|^{|\alpha|}$. Since $|\alpha| = k$ and $|x - x_0| < \rho$,

$$|(x - x_0)^\alpha| \leq \rho^k. \tag{6}$$

Putting together equations (3), (5) and (6), and using the multinomial theorem ($\sum_{|\alpha|=k} \frac{k!}{\alpha!} = n^k$), we obtain

$$|R_k(x)| \leq \frac{M}{e} \left(\frac{en\rho}{r}\right)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} = \frac{M}{e} \left(\frac{en^2\rho}{r}\right)^k.$$

Therefore R_k converges to zero uniformly for $x \in B(x_0; \rho)$ when k goes to infinity if $\frac{en^2\rho}{r} < 1$. So the Taylor series is convergent for any $x \in B(x_0; \rho)$ if $\rho < \frac{r}{en^2} < r$.

□