games of cooperation on graphs

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the paradox of cooperation

cost-benefit dilemma = prisoner’s dilemma → cooperation never survives evolution

however, cooperation is widespread in nature

what’s wrong?

EGT?

ways out of the conundrum:

- kin-selection
- direct reciprocity
- indirect reciprocity
- network reciprocity
Cooperation matters to

- Biology, Behaviour Ecology, Anthropology
- Economy, Sociology, Political Science
- Philosophy, Psychology, Theology, etc . . .

dealt with by

- Biologists, Ecologists, Anthropologists
- Economists, Sociologists, Political Scientists
- Philosophers, Psychologists,
- Mathematicians, Physicists, etc . . .

interdisciplinary ? YES !!!

with a common math background: game theory
finite populations
finite, well-mixed population
each individual interacts with ALL others

complete graph
finite & well-mixed pops

- finite populations → countable number of individuals;

- we cannot use anymore the “density of cooperators” \( x \), since now the change in the number of cooperators is not anymore continuous;

- the discrete nature of the population → stochastic description;

- single-round games → Markov process;

- Birth-Death processes: Markov processes which keep population size constant;

- what is the equivalent of the ESS condition in a finite population?
  
  **fixation probability**: probability that, introducing a mutant in a homogeneous population, its strategy invades the entire population.
stochastic model (birth-death but not Moran)

pairwise comparison interaction (also Fermi process)

at every “time”-step:
- randomly choose 2 individuals (A & B) from the population;
- strategy of B replaces that of A with a prob given by

\[ p = \left[ 1 + e^{-\beta(\Pi_B - \Pi_A)} \right]^{-1} \]

- otherwise, strategy of A replaces that of B;
- \( \Pi_A \) and \( \Pi_B \) are the payoffs of A and B;
- \( \beta \) controls how smoothly the probability changes from 0 to 1:
finite, complete graph

\[
\begin{align*}
\pi_C(l) &= R(l - 1) + S(N - l) \\
\pi_D(l) &= T \cdot l + P(N - l - 1)
\end{align*}
\]

prob a single C will invade a pop of Ds:

\[
\phi_1 = \left[ 1 + \sum_{k=1}^{N-1} \prod_{l=1}^{k} \frac{\pi_D(l)}{\pi_C(l)} \right]^{-1}
\]
pairwise comparison interaction

for given $N$ & $\beta$ the prob that $k$ Cs reach fixation is

$\phi_k = \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} \alpha_j}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} \alpha_j} \quad \alpha_j = \frac{T_j^-}{T_j^+}$

where:

$T_j^\pm = \frac{j}{N} \cdot \frac{N-j}{N} \cdot \frac{1}{1 + e^{\pm \beta(\pi_C(j) - \pi_D(j))}}$

prob select C

prob select D

take-over prob
pairwise comparison interaction

defining

\[ 2u = P + R - S - T \]
\[ 2v = P - R + N(S - P) \]

\[ \phi_k = \frac{\sum_{i=0}^{k-1} e^{-\beta_i(i+1)u - 2\beta iv}}{\sum_{i=0}^{N-1} e^{-\beta_i(i+1)u - 2\beta iv}} \]

since

\[ \sum_{i=0}^{k-1} e^{-\beta_i(i+1)u - 2\beta iv} = \frac{\sqrt{\pi}}{2} \left( \int_{\chi_0}^{\chi_k} dy \ e^{-y^2} - \int_{0}^{\chi_0} dy \ e^{-y^2} \right) + O(N^{-2}) \]
\[ \approx \frac{\sqrt{\pi}}{2} \left( \text{Erf} \ (\chi_k) - \text{Erf} \ (\chi_0) \right) \]

hence:
for given $N$ & $\beta$ the \textit{Prob} that $k$ \textit{Cs} reach fixation is

$$\phi_k = \frac{\text{Erf}(\chi_k) - \text{Erf}(\chi_0)}{\text{Erf}(\chi_N) - \text{Erf}(\chi_0)}$$

with

$$\chi_k = \sqrt{\frac{\beta}{u}}(ku + v)$$

where

whenever $\phi_k > \frac{k}{N}$ cooperation will be favoured !!!
results for fixation probabilities

Prisoner’s Dilemma
T > R > P > S

Stag-Hunt game
R > T > P > S

Snowdrift game
T > R > S > P

$k/N$

$\Phi_k$

$\beta = 0.05$

$\beta = 0.1$

$\beta = 0.1$

$\beta = 0.1$

$\beta = 0.1$

Traulsen, Nowak & JmP
PRE 74(2006) 011909

Moran
limit for large $N$

For large populations $\rightarrow$ stochastic diff. equation

( Traulsen, Claussen & Hauert *PRL* 95 (2005) 238701 )

with
diffusion term:

$$\frac{1}{N^2} \sqrt{T^+_j + T^-_j}$$

& drift term:

$$T^+_j - T^-_j$$

leading to the equation

$$\dot{x} = x(1-x) \tanh \left( \frac{\beta}{2} (\pi_C - \pi_D) \right) + \sqrt{\frac{x(1-x)}{N}} \xi$$

[ Traulsen, Nowak, JmP *PRE* 74(2006) 011909 ]

low-$\beta$, large-$N$ expansion:

$$\dot{x} = x(1-x)(\pi_C - \pi_D)$$

replicator equation !!!

white noise (gaussian) with variance 1.
what happens when the graph is *NOT* complete?

is there any change in the fixation probability?

is life *easier* for cooperators?
what happens when the graph is \textit{NOT} complete?  

is there any change in the fixation probability?  

is life \textit{easier} for cooperators?  

YES!
In ALL graphs, the average connectivity is $<k> = 4$
analytic results for static graphs

approximate results were obtained when:

- graphs are *static*
- graphs are (very) *large*
- graphs are homogeneous
- graphs have no loops
  - *trees*
  - *homogeneous random graphs*
- $\beta \ll 1$ ( weak selection limit )
- pair-approximation method
  ( correlations only between nearest neighbours )
results for static graphs

- fixation probability depends on evolutionary dynamics:
  - Moran Birth-death – \( \rho_1 < \frac{1}{N} \)
  - Moran death-Birth – \( \rho_1 > \frac{1}{N} \) if \( \frac{b}{c} > \langle k \rangle \)


numerical simulations show validity for smaller \( N \)!
what happens for other types of graphs?
numerical simulations on static graphs

replicator dynamics on graphs

numerical implementation:

@ every time-step:
choose 1 individual @ random (A)
compute A’s fitness and that of all A’s neighbours
everyone plays once with all neighbours &
individual fitness = accumulated payoff
choose a random neighbour of A (B)
compare fitness of A & B, \( \Pi_A \) & \( \Pi_B \)
if \( \Pi_A \geq \Pi_B \) nothing happens
otherwise B replaces A with probability

\[
p = \frac{\Pi_B - \Pi_A}{D_k}
\]

\[
D_k = \max(T, R) - \min(P, S)
\]

\[
k = \max(k_A, k_B)
\]

numerical technique:

starting from 50% Cs & 50% Ds
evolve population for a long time (til reaching stationary regime)
average fraction of Cs for another long time
repeat many times & average everything @ the end
numerical results for static graphs

[ Santos, JmP, Lenaerts, PNAS 103 (2006) 3490-3494 ]
numerical results for static graphs

the preferential attachment algorithm of Barabasi & Albert helps promoting cooperation; HOW?

[Santos, JmP, Lenaerts, PNAS103 (2006) 3490-3494]
what’s going on?

The larger the heterogeneity, the more chances do cooperators have to outcompete defectors.

**HOW?**

<table>
<thead>
<tr>
<th>type</th>
<th>heterogeneity</th>
</tr>
</thead>
<tbody>
<tr>
<td>regular</td>
<td>0</td>
</tr>
<tr>
<td>single-scale</td>
<td>moderate</td>
</tr>
<tr>
<td>scale-free</td>
<td>large</td>
</tr>
</tbody>
</table>

---


what's going on?

**heterogeneity** ➔ **cooperators** may outperform **defectors**;

although **defectors** have a “dilemma-advantage”, the **fitness of cooperators** may outweigh that of defectors if they interact more, that is, if **cooperators occupy nodes with higher degree**;
heterogeneity: a new route to cooperation

**Payoff Matrix:**

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3T+2P</td>
<td>5R+2S</td>
</tr>
</tbody>
</table>

**Fitness Functions:**

\[ f^{C_{HET}} = 5R + 2S \]
\[ f^{C_{HOM}} = 2R + 2S \]
\[ f^{D_{HET}} = 3T + 2P \]
\[ f^{D_{HOM}} = 2T + 2P \]

**Text:**

heterogeneity introduces a *connectivity-dependent term* in the fitness of each individual

**cooperators breed cooperators**: successful cooperators will reinforce a cooperative neighborhood → increase of fitness in next generations

**defectors breed defectors**: successful defectors will reinforce a defector neighborhood → decrease of fitness in next generations

**WHO WINS? COOPERATORS!!!**
heterogeneity: a new route to cooperation

if cooperators help cooperators to outperform defectors, direct connections between highly connected individuals should promote cooperation, since cooperators end up occupying the hubs.

but this is precisely what the Barabasi & Albert algorithm gives us for free!

why?

growth & preferential attachment build age-correlations: old nodes in generation time are the most connected.
growth
the older, the more connected

\[ m = m_0 = 2 \]

→ exponential \( d(k) \)

preferential attachment

\textbf{even more :}

the older, the more connected

→ power-law \( d(k) \)

\[
\begin{array}{c|c}
 n & K_n \\
\hline
 1 & 1 \\
 2 & 1 \\
\end{array}
\]
growth
the older, the more connected

\[ m = m_0 = 2 \]

exponential \( d(k) \)

preferential attachment

even more:
the older, the more connected

power-law \( d(k) \)

\[
\begin{array}{c|c}
 n & K_n \\
 1 & 2 \\
 2 & 2 \\
 3 & 2 \\
\end{array}
\]
designing cooperation

growth
the older, the more connected

\[ m = m_0 = 2 \]

exponential \( d(k) \)

preferential attachment

even more:
the older, the more connected

power-law \( d(k) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( K_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>
growth
the older, the more connected

\[ m = m_0 = 2 \]

→ exponential \( d(k) \)

preferential attachment

even more:
the older, the more connected

→ power-law \( d(k) \)
with the hubs directly connected, we can now form a *league of gentlemen* (cooperators) who help each other to sustain cooperation & outcompete defectors
league of gentlemen

generations: 0
what happens when we change $z$?
regular graph results

regular Ring-graphs:
regular graph results

ring-graphs: simulation results

c = 0, \ N = 10^4
regular graph results

ring-graphs: simulation results

\[ c = 0, \quad N = 10^4 \]
ring-graphs : simulation results

$c = 0, \ N = 10^4$
ring-graphs: simulation results

regular graph results

c = 0, N = 10^4

![Graph showing fraction of cooperators vs. b](image-url)
ring-graphs : simulation results

regular graph results

c = 0, N = 10^4
summary of results for static graphs

replicator dynamics on graphs

on homogeneous graphs cooperators have little chances in the benefit-cost PD;

heterogeneity opens a new route for the evolution of cooperation;

the more the most connected individuals are directly connected to each-other, the more cooperation is self-sustained;

cooperators have a chance only when networks are sparse; high connectivities are unfavourable to cooperation

\[
\frac{b}{c} \rightarrow z
\]

makes sense . . .
individuals engage in more than one game \((w<<1)\)

network of interactions is often more complex!

individuals often use more than one network

networks are not static
breaking the symmetry between interaction & replacement
breaking the symmetry between *interaction* & *replacement*

1 population!  
several nets!

*Interaction graph:*  
business, social interactions, etc.

*replacement graph:*  
reproduction, imitation, learning, etc.
We define:

- **interaction graph** \((H,h)\):
  - “who plays with who”

- **replacement graph** \((G,g)\):
  - “evolutionary competition & updating”

- **technique**
  - pair-approximation method

- **evolutionary dynamics**
  - death-Birth

[ JmP, Ohtsuki, Nowak, *PRL* 98 (2007) 108106 ]
[ Ohtsuki, JmP, Nowak, *JTB* 246 (2007) 681 ]
cooperation is most favoured when interaction and replacement graphs coincide !!!!
features of approaches up to now

individuals have no control
- whom they interact with;
- duration and frequency of interactions;

cooperation is feasible only on sparse graphs strongly heterogeneous graphs

but social networks are
- single to broad scale;
- NOT sparse ($<k> \sim 30$);
- constantly evolving;
co-evolution of strategy & structure
If you have a well-defined behaviour: C or D

what is your best (most convenient) partner?

For all social dilemmas

The best partner for any strategy is always a Cooperator

Consequently, irrespective of the dilemma:

Cs look for Cs to cooperate with
Ds look for Cs to exploit
choose A @ random; then choose a neighbour B of A @ random;

B is satisfied (A is C); A is NOT satisfied (B is D);

B wants to keep link; A wants to change;

with probability p A rewires to a neighbour of B;

with probability (1-p) B keeps link;

remember: a neighbour of a defector is “most likely” a cooperator;
co-evolution of strategy & structure

now we have two different time-scales:

- strategy evolution $T_S$
- structural evolution $T_A$

we define the ratio $W = \frac{T_S}{T_A}$

*and study co-evolution as a function of $W$*

for ALL social dilemmas;

features of the model:

- population is constant
- average connectivity remains constant
- graph remains connected
co-evolution of strategy & structure

A wave of cooperation moves south-east... as rewiring dynamics becomes faster
co-evolution of strategy & structure

\[ \frac{b}{c} = 2 \]

The larger \( z \), the harder it gets

Heterogeneity helps cooperation

\[ W_{\text{crit}} \uparrow \rightarrow k_{\text{max}} \uparrow \]

Average connectivity
co-evolution of strategy & structure

PD

PD: heterogeneity is maximal @ $W_{\text{crit}}$...
co-evolution of strategy & structure

- **W = 1.0**: No need for heterogeneity; everybody cooperates.
- **W = 2.0**: Greed promotes heterogeneity.
- **W = 3.0**: Fear promotes clustering.
- **W = 4.0**: Heterogeneity helps cooperation.

Greed promotes heterogeneity.
co-evolution of strategy & structure

can we understand these results analytically?
populations as dynamical graphs

a simple, analytical model . . .

[ JmP, Traulsen, Nowak, PRL 97 (2006) 258103 ]
[ JmP, Traulsen, Nowak, JTB 243 (2006) 437 ]
C → $\alpha_C$ → propensity to form a new link
D → $\alpha_D$ → propensity to form a new link

$X$ → CC-links → lifetime $\tau_{CC}$ → death rate $\gamma_{CC} = \tau_{CC}^{-1}$
$Y$ → CD-links → lifetime $\tau_{CD}$ → death rate $\gamma_{CD} = \tau_{CD}^{-1}$
$Z$ → DD-links → lifetime $\tau_{DD}$ → death rate $\gamma_{DD} = \tau_{DD}^{-1}$
\( N \) Individuals : \( l \) cooperators & \( N - l \) defectors

**CC**
\[ X_m = \frac{1}{2} l(l - 1) \]

**CD**
\[ Y_m = l(N - l) \]

**DD**
\[ Z_m = \frac{1}{2} (N - l)(N - l - 1) \]  
(max links)

**Active Linking Dynamics :**

\[
\dot{X} = \alpha_C^2 (X_m - X) - \gamma_{CC} X \\
\dot{Y} = \alpha_C \alpha_D (Y_m - Y) - \gamma_{CD} Y \\
\dot{Z} = \alpha_D^2 (Z_m - Z) - \gamma_{DD} Z
\]
steady state:

\[
X^* = X_m \frac{\alpha_C^2}{\alpha_C^2 + \gamma_{CC}} = X_m \eta_{CC}
\]

\[
Y^* = Y_m \frac{\alpha_C \alpha_D}{\alpha_C \alpha_D + \gamma_{CD}} = Y_m \eta_{CD}
\]

\[
Z^* = Z_m \frac{\alpha_D^2}{\alpha_D^2 + \gamma_{DD}} = Z_m \eta_{DD}
\]

payoffs of Cs & Ds:

\[
c_l = R \eta_{CC} (l - 1) + S \eta_{CD} (N - l)
\]

\[
d_l = T \eta_{CD} l + P \eta_{DD} (N - l - 1)
\]
Active Linking Dynamics (ALD)

$N = 40$

$l = 10, 20, 30$

$\alpha_C = \alpha_D = 0.4$

$\gamma_{CC} = 0.16$

$\gamma_{DD} = 0.32$

$\gamma_{CD} = 0.80$

Because ALD depends on the actual frequency of cooperators, there is a natural coupling between structure and dynamics.

[ JmP, Traulsen, Nowak JTB 243 (2006) 437 ]
Active Linking Dynamics (ALD)

finite, well-mixed pops

\[ c_l = R(l - 1) + S(N - l) \]
\[ d_l = Tl + P(N - l - 1) \]

finite pops in steady-state ALD

\[ c_l = R'(l - 1) + S'(N - l) \]
\[ d_l = T'l + P'(N - l - 1) \]

evolutionary dynamics in steady state ALD is equivalent to that in finite well-mixed populations for a \textit{scaled payoff matrix}

\[
\begin{bmatrix}
  R & S \\
  T & P \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  R' & S' \\
  T' & P' \\
\end{bmatrix}
\equiv
\begin{bmatrix}
  R \eta_{CC} & S \eta_{CD} \\
  T \eta_{CD} & P \eta_{DD} \\
\end{bmatrix}
\]

but . . .
Active Linking Dynamics

we have **two** time scales:
- rewiring time scale ($T_A$);
- strategy evolution time scale ($T_S$);

**strategy evolution**: pairwise comparison

$$p(A \rightarrow B) = \left[1 + e^{-\beta(\Pi_B - \Pi_A)}\right]^{-1}$$

for $\beta << 1$: pairwise comparison =

= Birth-death

analytical results for the fixation probability of an arbitrary number of Cs

[ Traulsen, Nowak & JmP _PRE_ 74(2006) 011909 ]
[ Traulsen, Nowak & JmP _JTB_ 246 (2007) 522 ]
Active Linking Dynamics

**\( T_A << T_S \):** pop reaches ALD-steady state before next strategy change

whenever \( \phi_k > \frac{k}{N} \)

cooperation will be favoured !!!

\[
\phi_k = \frac{Erf(\chi_k) - Erf(\chi_0)}{Erf(\chi_N) - Erf(\chi_0)}
\]

\[
\chi_k = \sqrt{\frac{\beta}{u}} (ku + v)
\]

**\( 2u = R' + P' - S' - T' \)**  
**\( 2v = -(N-1)P' \)**

**\( T_A >> T_S \):** pop evolves without ANY ALD-update

whenever \( \phi_k > \frac{k}{N} \)

cooperation will be favoured !!!

\[
\phi_k = \frac{Erf(\chi_k) - Erf(\chi_0)}{Erf(\chi_N) - Erf(\chi_0)}
\]

\[
\chi_k = \sqrt{\frac{\beta}{u}} (ku + v)
\]

**\( 2u = R + P - S - T \)**  
**\( 2v = -(R - N)(S - (N-1)P) \)**
co-evolutionary dynamics

example 1:

Snowdrift game \rightarrow Harmony game

\[ \beta = 0.1 \quad N = 100 \]
\[ \alpha_C = \alpha_D = 0.4 \]
\[ \gamma_{CC} = 0.1 \quad \gamma_{CD} = 0.8 \quad \gamma_{DD} = 0.32 \]
\[ b = 1.0 \quad c = 0.8 \]

[ JmP, Traulsen, Nowak, PRL 97 (2006) 258103 ]
example 2:

Prisoner’s dilemma → Coordination game

\[
\begin{align*}
\beta &= 0.1 \\
N &= 100 \\
\alpha_C &= \alpha_D = 0.4 \\
\gamma_{CC} &= 0.1 \\
\gamma_{CD} &= 0.8 \\
\gamma_{DD} &= 0.32 \\
b &= 1.0 \\
c &= 0.5
\end{align*}
\]

[ JmP, Traulsen, Nowak, PRL 97 (2006) 258103 ]
ex. 2 (contd) :

whenever \( T_A \sim T_S \) computer simulations:

- For \( T_S/T_A \sim 0.1 \) the PD has been effectively transformed into a coordination game – ALD has a much stronger impact than static graphs in the EOC.

\[ C_{\text{start}} = 50\% \]
\[ D_{\text{start}} = 50\% \]

\[ \begin{array}{cc}
C & D \\
C & 4 & 2 \\
D & 5 & 3 \\
\end{array} \]

\[ \begin{array}{cc}
C' & D' \\
C' & 2 & 1/3 \\
D' & 5/6 & 1 \\
\end{array} \]

\[ \beta = 20.0 \]
\[ \beta = 0.05 \]

[ JmP, Traulsen, Nowak, PRL 97 (2006) 258103 ]
co-evolutionary dynamics

re-discovering Hamilton’s rule:

define advantage of cooperators from ALD

\[ r = \frac{\eta_{CC} - \eta_{CD}}{\eta_{CC}} \]

\[
\begin{array}{c|cc}
SG & C & D \\
\hline
C & b - \frac{c}{2} & b-c \\
D & b & 0 \\
\end{array}
\]

SG → Harmony game

PD → Coordination game

co-evolutionary dynamics

introducing (a simple) game feedback mechanism onto graph dynamics:

\[ \alpha_c = \alpha_D = \alpha \]

\[ \tau_{ij} = \frac{\kappa}{2}(a_{ij} + a_{ji}) \]

\[ \theta = \alpha^2 \tau \]

\[ r = \frac{p-1}{p+\theta} \]

\[ \tau_{CD} = \tau_{CC} / 2 \rightarrow \tau_{CC} / p, p > 1 \]
few & heterogeneous interaction patterns help cooperators to outcompete defectors

on static graphs cooperation is maximized whenever those one interacts with are also our role models

cooperrators are able to outcompete defectors whenever they react quickly enough to adverse ties (CD), resulting in the assortative linking among Cs

population structure acts to effectively change the “nature of the game”
END